2023 Leo Schneider Student Team Competition

1. Take an integer greater than 9. Double its last digit and subtract this from the number formed by the remaining digits. Prove that this difference is divisible by 7 if and only if the original number is divisible by 7. For example, if our original number is 2023, we get $202 - 6 = 196$, and repeating this process gives us $19 - 12 = 7$, which is divisible by 7. Therefore, 196, and hence 2023, is divisible by 7 (indeed $2023 = 7 \cdot 289$). On the other hand, if our original number is 2024, we get $202 - 8 = 194$, and repeating this process gives us $19 - 8 = 11$, which is not divisible by 7.

2. Let ABC be a triangle in which $m\angle B$ and $m\angle C$ are both greater than 60°. Let BCD be an equilateral triangle, with D inside $\triangle ABC$, and let BEF be another equilateral triangle, with E on AB and F on AC. Suppose, moreover, that triangles BCD and BEF are congruent. Show that $m\angle A < 30^\circ$.

3. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible? Prove your answer.

4. Find all right triangles with integer side lengths such that the area and perimeter of the triangle evaluate to the same number (with different units, obviously!).

5. Find a set of three consecutive odd integers $\{a, b, c\}$ for which the sum of squares $a^2 + b^2 + c^2$ is an integer made of four identical digits. For example, 2222 is an integer made of four identical digits and $\{7, 9, 11\}$ is a set of three consecutive odd integers.

6. A tangent line to the ellipse $x^2 + 4y^2 = 4$ meets the x-axis and y-axis at the points A and B, respectively. Find the minimum value of AB, the length of the line segment with endpoints A and B.

7. Find $\int_0^1 x \arcsin x \, dx$.

8. In a magical isosceles triangle $\triangle ABC$, we have $|AC| = |BC|$. Let D be the midpoint between A and B. The inscribed circle of $\triangle ABC$ intersects the line segment CD in a point E that is in the interior of the triangle. Suppose that $|AB| = 15$ and $|CE| = 8$. Determine $|AC|$.

9. Determine the number of ways that one can tile a 4×13 rectangle with 4×1 rectangles.

10. Katie and David are playing a game in which Katie rolls a fair n-sided die and David rolls a fair m-sided die. Katie rolls first and the winner is the first person to roll a 1. Katie and David are equally likely to win the game. How are m and n related?

2022 Leo Schneider Student Team Competition

1. Find (with proof) all whole number values of n so that the binary expansion of $n!$ contains exactly two 1's.

2. Consider a game played on a finite sequence of positive integers in which two types of moves, A and B, are allowed: A move of type A ("Add") replaces two adjacent integers in the sequence by their sum; for example $(\ldots, 20, 11, \ldots) \xrightarrow{A} (\ldots, 31, \ldots)$. A move of type B ("Break up") replaces a multi-digit integer in the sequence by the sequence of its *nonzero* decimal digits; for example, $(\ldots, 2021, \ldots) \xrightarrow{B} (\ldots, 2, 1, 1, \ldots)$.
   The moves may be combined in any manner. For example, given the sequence $(3, 14, 159, 26)$, a possible sequence of moves is the following:

$$(3, 14, 159, 26) \xrightarrow{A} (17, 159, 26) \xrightarrow{B} (17, 1, 5, 9, 26) \xrightarrow{A} (18, 5, 9, 26) \xrightarrow{A} (18, 5, 35)$$
$$\xrightarrow{A} (18, 40) \xrightarrow{B} (18, 4) \xrightarrow{B} (1, 8, 4) \xrightarrow{A} (9, 4) \xrightarrow{A} (13) \xrightarrow{B} (1, 3) \xrightarrow{A} (4)$$

   Once the sequence is reduced to a single one-digit number, any further moves will leave it unchanged, the game terminates, and we call the final number obtained the *terminal number* of the game. Suppose this game is played on the sequence $(1, 2, 3, 4, \ldots, 2022)$. What is the terminal number? (For full credit, you must also show that this number is unique.)

3. Let $f(x) = x^2 + bx + c$ where b and c are real numbers between -1 and 1, selected uniformly at random. What is the probability that both roots of $f(x)$ are positive real numbers?

4. Suppose that A and B are two $n \times n$ matrices. Prove that

$$(A + AB^{-1}A)^{-1} + (A + B)^{-1} = A^{-1},$$

   assuming that all these inverses exist.

5. An urn contains 3 red balls and 2 blue balls. A second urn contains 2 red balls and an unknown number of blue balls. Two balls are drawn at random from the first urn and placed in the 2nd urn. A ball is then randomly drawn from the 2nd urn and the probability that it is blue is $\frac{4}{5}$. How many blue balls are initially in the 2nd urn?

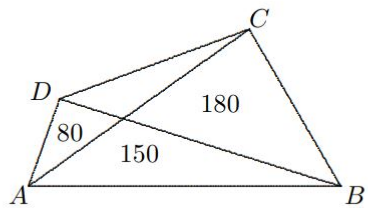
6. Given an equilateral triangle $\triangle ABC$ with sides of length 1 unit and with an arbitrary interior point P, draw the line segments $\overline{\mathrm{PD}}$, $\overline{\mathrm{PE}}$ and $\overline{\mathrm{PF}}$ that are perpendicular to the 3 sides of the triangle. Let $s = PD + PE + PF$, the sum of the lengths of the 3 segments. Find the minimal value of s.

7. Find $\int \ln(x + \sqrt{x^2 - 1})\, dx$.

8. Show that 2022 divides $676^{10} - 1024$.

9. In polar coordinates, $r = 1 + \cos\theta$ describes a cardioid. Find the maximum (vertical) distance between this cardioid and the x-axis.

10. The quadrilateral $ABCD$ is partitioned into four triangles by means of the diagonals AC and BD. The areas of three of the triangles are indicated in the figure below.



What is the area of the quadrilateral?

~~2020~~ 2021 Leo Schneider Student Team Competition

1. Find the last two digits of $2021! + 2020! + 2019! + \cdots + 2! + 1!$.

2. Find $\int_{-1}^{8} \sqrt{1+\sqrt{1+x}}\,dx$.

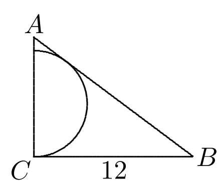
3. How many sequences of length 10 consisting of letters drawn with replacement from the set {M,A,T,H} have the letters in alphabetic order?

4. Find the sum of the digits of the number $9 + 99 + 999 + \cdots + 999\cdots 9$, where the last number in our sum consists of 2021 9s.

5. The matrix $M = \begin{bmatrix} t & 1-t \\ 1 & 2t \end{bmatrix}$ has 1 as an eigenvalue. What are the possibilities for the other eigenvalue?

6. Suppose a and b are real numbers such that
$$\lim_{x\to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} = \frac{1}{2}.$$
Determine all possible ordered pairs (a, b).

7. Consider the following right triangle, with $m(BC) = 12$:



A semicircle is drawn with a diameter along side AC. One endpoint of the diameter is C, and the other is exactly one unit away from A. If the semicircle is tangent to side AB, find the radius of the circle.

8. Let $f(x) = x\cos x$. Find, with proof, $f^{(2021)}(0)$, the 2021st derivative of f evaluated at 0.

9. Let $P(x)$ be a quadratic polynomial with real coefficients such that $P(3) = 2021$ and
$$P(x) = P(0) + P(1)x + P(2)x^2$$
for all real x. What is $P(-1)$?

10. You roll three fair six-sided dice. Find the probability that the product of the three numbers rolled is a perfect square.

2019 Leo Schneider Student Team Competition

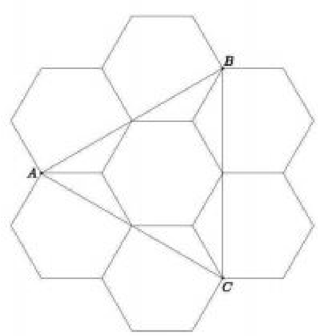
1. For each $a < 0$, consider the line perpendicular to the curve $y = x^2$ at $x = a$. This line intersects the curve at another point. Find the minimum possible value of the x-coordinate of this second intersection point.

2. Compute $\displaystyle\int \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx$.

3. Find the value of $\displaystyle\int_0^\infty \left(\frac{3}{4}\right)^{\lfloor x \rfloor} dx$, where $\lfloor x \rfloor$ is the **floor function**, which gives the largest integer less than or equal to x.

4. Three people, named A, B and C, throw a die alternately. First A throws, then B, then C, and this keeps repeating. What is the probability that A throws the first six, B the second six and C the third six?

5. Six regular hexagons surround a regular hexagon of side length 1 as shown. What is the area of $\triangle ABC$?



6. Find $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{\sqrt{n^2 + k}}$.

7. A two-digit number is uniquely determined by the answers to the following yes/no questions.
   1. Is the number divisible by 2?
   2. Is the number divisible by 3?
   3. Is the number divisible by 5?
   4. Is the number divisible by 7?

   If the answer to the first question is "yes", what is the number?

8. Write 2019 as the difference of squares of positive integers in two different ways.

9. What is the largest integer a such that 7^a divides $1000!$?

10. The base 5 representation of a positive integer has 3 digits. Reversing those digits gives the base 7 representation of the same number. What are the possible base 10 representations of that number?

2018 Leo Schneider Student Team Competition

1. Find $\int \frac{x^5}{(x^3+1)^2} dx$.

2. Evaluate the following limit:

$$\lim_{x\to 0} \frac{\sin \arctan x - \tan \arcsin x}{\arcsin \tan x - \arctan \sin x}$$

3. Let $\left(1+\sqrt{2}\right)^n = A_n + B_n\sqrt{2}$ with A_n and B_n rational numbers.

   (a) Express $\left(1-\sqrt{2}\right)^n$ in terms of A_n and B_n.

   (b) Compute $\lim\limits_{n\to\infty} \frac{A_n}{B_n}$.

4. You roll three fair dice. Find the probability that some subset of the numbers you rolled sums to 3. (This includes possible rolls such as 1,5,3.)

5. What is the probability of an odd number of sixes turning up in a random toss of n fair dice?

6. $\triangle ABC$ satisfies $AC = 1$ and $\angle ACB$ is right. Points D and E are on $\overline{AB}$ with $\overline{CD}$ and $\overline{CE}$ trisecting $\angle ACB$. Suppose, moreover, that AB is the smallest integer such that AD, DE, and EB are distinct rational numbers. Find AB.

7. Let $A = \begin{bmatrix} -1 & 2018 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Compute $A^{2018}\mathbf{u}$.

8. Find all integers c such that x^2+c divides x^3+4x^2-x-4. Explain why you have found all the answers!

9. Given that $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x, find the following integral:

$$\int_0^2 \lfloor x \rfloor - 2\left\lfloor \frac{x}{2} \right\rfloor \, dx$$

10. Integers x, y, and z satisfy $x+2y+3z=60$, $4x+5y=60$, and $z \geq 0$. What is the maximum value of the product xyz?

2017 Leo Schneider Student Team Competition

1. Find
$$\int_4^\infty \frac{1}{\sqrt{x}(x-1)}\,dx.$$

2. If
$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots, v = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots, w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots,$$
prove that
$$u^3 + v^3 + w^3 - 3uvw = 1.$$

3. Find $\displaystyle \lim_{n\to\infty} \frac{\binom{2(n+1)}{n}}{\binom{2n}{n+1}}$.

4. Jason has an 8 a.m. class this semester, and he is not a morning person. If he wakes up before 7:30, he showers in the morning 60% of the time, but if he wakes up after 7:30, he showers only 10% of the time. He only manages to wake up before 7:30 25% of the time, but when he does, he makes it to class on time 80% of the time. When he doesn't wake up before 7:30, he only makes it to class on time 30% of the time. Assuming that on a given morning he takes a shower, what is the probability that he makes it to class on time? **Note.** Thanks to Professor Larry Robinson at ONU for pointing out a flaw in the problem.

5. Two points are chosen at random (with a uniform distribution) from the unit interval $[0, 1]$. What is the probability that the points will be within a distance of $1/8$ of each other?

6. Michael Phelps' favorite hobby in his offtime is alchemy. He has a machine that can change 1 gold medal into 2 silver and 3 bronze, another machine that can turn 1 silver medal into 2 gold and 1 bronze, and a third machine that can turn 2 bronze medals into 3 gold and 1 silver. Starting with the 23 gold medals, 3 silver medals, and 2 bronze medals that he already has, can he use his machines to end up with a total of exactly 27 gold medals, 17 silvers, and 18 bronzes? Find a sequence of uses of the machines that accomplishes this or explain why it can't be done.

7. Ten equally spaced points are marked around the circumference of a circle. Consider connecting them into decagons two ways: first, by connecting all the nearest pairs of points, and second, by connecting every third point.

Prove that the difference in side lengths of these two decagons is equal to the radius of the circle.

8. Find the largest integer that divides $p^2 - 1$ for all primes $p > 3$.

9. What is the smallest number divisible by 225 that (written in base 10) contains only the digits 1 and 0? Explain your answer.

10. A sequence begins with a_1, a_2, and for $n > 2$ is defined by $a_n = a_{n-1} - a_{n-2}$. Find the sum of the first 2016 terms and defend your answer.

2016 Leo Schneider Student Team Competition

1. Neither *Mathematica* nor *Maple* can find the exact value of the following definite integral. Can you? We think you can. Do it!

$$\int_0^2 (3x^2 - 3x + 1)\cos(x^3 - 3x^2 + 4x - 2)\,dx.$$

2. In a certain group of cancer patients, each patient's cancer is classified in exactly one of the following five stages: stage 0, stage 1, stage 2, stage 3, or stage 4.

   (a) 75% of the patients in the group have stage 2 or lower

   (b) 80% of the patients in the group have stage 1 or higher

   (c) 80% of the patients in the group have stage 0,1, 3, or 4.

   One patient is randomly selected. Calculate the probability that the selected patient's cancer is stage 1.

3. Compute

$$\int_0^{\sqrt{\pi/3}} \sin x^2 \; dx + \int_{-\sqrt{\pi/3}}^{\sqrt{\pi/3}} x^2 \cos x^2 \; dx.$$

4. Let $a_1, a_2, a_3, \cdots$ be the increasing sequence of positive integers that are divisible by 2 or 5. The sequence begins 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, ... Compute the sum of the following series.

$$\sum_{n=1}^{\infty} \frac{1}{2^{a_n}} = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \cdots$$

5. Find all integer solutions (x, y) to the equation $xy = 5x + 11y$.

6. Find

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n}} \right).$$

7. Let

$$f(r) = \sum_{j=2}^{2016} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{2016^r}.$$

   Find

$$\sum_{k=2}^{\infty} f(k).$$

8. Urn 1 contains 10 balls: 4 red and 6 blue. A second urn originally contains 15 red balls and an unknown number of blue balls. Joe draws a single ball from the first urn and places it into the second urn. He then draws a ball from the second urn. Prior to conducting the experiment, the probability that both balls drawn will be of the same color is 0.5. What is the probability the ball drawn from urn 1 was red given that the ball drawn from urn 2 was red?

9. The rhombicosidodecahedron is an Archimedean solid with faces that are equilateral triangles, squares, and regular pentagons. It has 60 vertices and 120 edges. If 20 of the faces are triangles and 12 are pentagons, how many must be squares?

10. Prove that $AB - BA \neq I_n$ for any $n \times n$ matrices A and B over the real numbers, where I_n denotes the $n \times n$ identity matrix.

2015 Leo Schneider Student Team Competition

1. Find $\int_0^\infty \frac{1}{\sqrt{e^x - 1}}\,dx$.

2. A fair die and an unfair die are in a bag. The probability of rolling a six with the unfair die is 1/4. One of the dice is randomly drawn from the bag (each one is equally likely to be chosen). A six is rolled with this die. You roll this same die again. What is the probability that a six is rolled?

3. Tom and Jerry decide today that they will not pick on each other on the n-th day if $a_n = -1$ where $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, and a_n for $n > 3$ is inductively given by $a_n = a_{n-1}a_{n-3}$. Will they pick on each other on the 1776-th day? Prove your answer.

4. For each positive integer k, let
$$A_k = \begin{pmatrix} 1 & k & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
   (a) Find a closed form expression for the matrix A_1^n in terms of n. Prove your answer.
   (b) Find all ordered pairs (k, n) of positive integers for which $A_k^n = A_{75}$.

5. Find the limit
$$\lim_{n\to\infty}\left[\frac{(1+\frac{1}{n})^n}{e}\right]^n$$

6. How many solutions does the equation $20x + 15y = 2015$ have over the positive integers?

7. A non-empty set of positive integers is said to be square-valued if the product of all of its elements is a perfect square. How many (non-empty) subsets of $\{10, 22, 30, 42, 55, 231\}$ are square-valued?

8. For the function, $f(x) = \ln\left(1 - \dfrac{1}{x^2}\right)$, find the value of:
$$f'(2) + f'(3) + f'(4) + \cdots + f'(2015).$$

9. Let S be the set of vertices of a regular 36-gon. What is the smallest value of n for which any subset of S of size n must contain the three vertices of some equilateral triangle? You must prove your answer.

10. Twenty calculus students are comparing grades on their first two quizzes of the year. The class discovers that whenever any pair of students consult with one another, these two students received the same grade on their first quiz or they received the same grade on their second quiz (or both). Prove that the entire class received the same grade on at least one of the two quizzes.

2014 Leo Schneider Student Team Competition

1. Find an equation with integral coefficients whose roots include the numbers
   (a) $\sqrt{2} + \sqrt{3}$
   (b) $\sqrt{2} + \sqrt[3]{3}$

2. Let a_1, a_2, $a_3, \cdots$ be the increasing sequence of positive integers that are not divisible by 2 or 3. The sequence begins 1, 5, 7, 11, 13, 17, $\cdots$ Compute the sum of the following series.
$$\sum_{n=1}^{\infty} \frac{1}{2^{a_n}} = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \cdots$$

3. It is well-known that
$$\int_0^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$
   Find
$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$

4. Evaluate
$$\lim_{x\to\infty} \frac{\int_0^x \sqrt{4+t^4}\, dt}{x^3}.$$

5. Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n.

6. Choose N elements of $\{1, 2, 3, \ldots, 2N\}$ and arrange them in increasing order. Arrange the remaining N elements in decreasing order. Let D_i be the absolute value of the difference of the ith elements in each arrangement. Prove that $D_1 + D_2 + \cdots + D_N = N^2$.

7. If n is a positive integer, let $r(n)$ denote the number obtained by reversing the order of the decimal digits of n. For example, $r(382) = 283$ and $r(410) = 14$. For how many two digit positive integers n is the sum of n and $r(n)$ a perfect square?

8. Determine the number of three word phrases that can be formed from the letters in MATH ALL DAY. No "words" can be empty, and words do not have to make sense. For example, MAD HAT ALLY and T DMALL YAAH are valid phrases, but not HALT MALADY. You do not have to simplify your answer.

9. A circle of radius r is inscribed in a right triangle with leg $4r$. Prove that the triangle is a 3-4-5 right triangle.

10. Recall that the set of real numbers forms an infinite-dimensional vector space V over the rationals (i.e., the set of vectors are the real numbers, and the set of scalars are the rational numbers). Let $2, 3, 5, 7, \ldots, p_{2014}$ be the first 2014 prime numbers, and define the vectors $v_k = \ln(p_k)$ for $k = 1, 2, \ldots, 2014$. Let S be the subspace of V spanned by the vectors $v_1, v_2, \ldots, v_{2014}$. Find, with proof, the dimension of S.

2013 Leo Schneider Student Team Competition

1. For the arithmetic sequence $a_1, a_2, ..., a_{16}$, it is known that $a_7 + a_9 = a_{16}$. Find each subsequence of three terms that forms a geometric sequence.

2. Compute the limit:
$$\lim_{x\to\infty} \frac{1}{xe^x} \int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt.$$

3. Part a. Let $f(x) = e^x \sin x$. Find $f^{(10)}(0)$, the 10th derivative of f evaluated at $x = 0$.

   Part b. Let $f(x) = e^x \sin x$. Find $f^{(2013)}(0)$, the 2013th derivative of f evaluated at $x = 0$.

4. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \le 13x^2$.

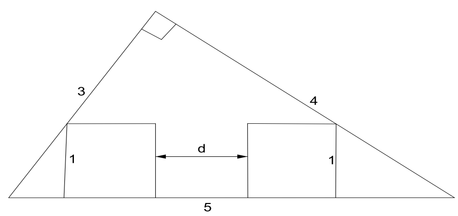
5. Let N be a positive integer containing exactly 2013 digits none of whose digits is zero. Show that N is either divisible by 2012 or N can be changed to an integer that is divisible by 2013 by replacing some but not all of its digits by zero.

6. Prove that $2^{2013} + 3$ is a multiple of 11.

7. A random number generator randomly generates integers from the set $\{1, 2, \ldots, 9\}$ with equal probability. Find the probability (with explanation) that after n numbers are generated, their product is a multiple of 10.

8. Planet A is going to launch n missiles at Planet B, which has n cities. Each missile will hit exactly one city. For each missile, the Planet B city that gets hit is completely random. Find the probability that exactly one city on Planet B will not get hit with any of the n missiles.

9. Two unit squares stand on the hypotenuse of a (3,4,5) triangle in such a way that they line inside the triangle, and a corner of one touches the side of length 3 and a corner of the other touches the side of length 4, as shown in the figure to the right. What is the distance d between the squares?



10. Let A and B be 3×3 matrices with integer entries, such that $AB = A + B$. Find all possible values of $\det(A - I)$. Note: The symbol I represents the 3×3 identity matrix.