2023 Leo Schneider Student Team Competition (with solutions)

1. Take an integer greater than 9. Double its last digit and subtract this from the number formed by the remaining digits. Prove that this difference is divisible by 7 if and only if the original number is divisible by 7. For example, if our original number is 2023, we get $202 - 6 = 196$, and repeating this process gives us $19 - 12 = 7$, which is divisible by 7. Therefore, 196, and hence 2023, is divisible by 7 (indeed $2023 = 7 \cdot 289$). On the other hand, if our original number is 2024, we get $202 - 8 = 194$, and repeating this process gives us $19 - 8 = 11$, which is not divisible by 7.

   Solution. Let $N = a_n 10^n + a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$, where $n > 0$, $0 \leq a_i \leq 9$ for all i, and $a_n \neq 0$. Put $\overline{N} = a_n 10^{n-1} + a_{n-1}10^{n-2} + \cdots + a_1 - 2a_0$. Notice that $10\overline{N} = N - 21a_0$. Therefore, $3\overline{N} \equiv N \pmod 7$. Since 3 is relatively prime to 7, we have that N is divisible by 7 if and only if $\overline{N}$ is divisible by 7.

2. Let ABC be a triangle in which $m\angle B$ and $m\angle C$ are both greater than $60°$. Let BCD be an equilateral triangle, with D inside $\triangle ABC$, and let BEF be another equilateral triangle, with E on AB and F on AC. Suppose, moreover, that triangles BCD and BEF are congruent. Show that $m\angle A < 30°$.

   Solution. Let $\theta = m\angle A$. Since $BF = BC$, $\triangle BCF$ is an isosceles triangle and thus

$$\begin{aligned} m\angle ACB &= m\angle BFC \\ &= 180° - m\angle BFA \\ &= 180° - (180° - \theta - m\angle ABF) \\ &= \theta + 60° \end{aligned}$$

   Therefore, $m\angle ABC = 180° - (\theta + 60°) - \theta = 120° - 2\theta$. Since $m\angle ABC > 60°$, this forces $\theta < 30°$.

3. Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible? Prove your answer.

   Solution. $A^2 + B^2$ cannot be invertible.

   Assume, by contradiction that A and B are different $n \times n$ matrices with real entries and that $A^2 + B^2$ is invertible, ie. $(A^2 + B^2)^{-1}$ exists. Then,

$$\begin{aligned} (A^2 + B^2)A &= A^3 + B^2A \\ &= B^3 + B^2A \\ &= B^3 + A^2B \\ &= (B^2 + A^2)B \\ &= (A^2 + B^2)B \end{aligned}$$

   Multiplying both sides on the left by $(A^2 + B^2)^{-1}$ yields $A = B$. But A and B were different, which is a contradiction.

4. Find all right triangles with integer side lengths such that the area and perimeter of the triangle evaluate to the same number (with different units, obviously!).

   Solution. Suppose the legs have length a and b. Then we wish to solve the equation

$$\frac{1}{2}ab = a + b + \sqrt{a^2 + b^2}$$

over the positive integers. Isolating the radical, squaring both sides, rearranging terms, and factoring gives us

$$ab(ab - 4a - 4b + 8) = 0,$$

which means that $a = 4(b-2)/(b-4)$. Put $u = b - 4$. Then we have that $a = 4 + 8/u$. Since a and b are positive integers, the only possible values for u are 1, 2, 4, and 8. So the only triangles that satisfy our hypotheses are 5-12-13 and 6-8-10 triangles.

5. Find a set of three consecutive odd integers $\{a, b, c\}$ for which the sum of squares $a^2 + b^2 + c^2$ is an integer made of four identical digits. For example, 2222 is an integer made of four identical digits and $\{7, 9, 11\}$ is a set of three consecutive odd integers.

Solution. We claim the set $\{41, 43, 45\}$ satisfies the conditions. To show this, consider three consecutive odd integers, $2n-1$, $2n+1$, and $2n+3$. Let the sum of the three squares be $1111x$ where x is the repeating digit, $1 \le x \le 9$. We have,

$$\begin{aligned}
(2n-1)^2 + (2n+1)^2 + (2n+3)^2 &= 1111x \\
(4n^2 - 4n + 1) + (4n^2 + 4n + 1) + (4n^2 + 12n + 9) &= 1111x \\
12n^2 + 12n + 11 &= 1111x
\end{aligned}$$

Taking mod 2, we have $x \equiv 1 \bmod 2$, so x must be odd.
Taking mod 3, we have $x \equiv 2 \bmod 3$, so $x = 2, 5$ or 8.
This means x must be 5 and $12n^2 + 12n + 11 = 5555$. Solving this equation gives,

$$\begin{aligned}
12n^2 + 12n + 11 &= 5555 \\
12n^2 + 12n - 5544 &= 0 \\
n^2 + n - 462 &= 0 \\
(n-21)(n+22) &= 0
\end{aligned}$$

Since n must be odd, $n = 21$ and thus $2n - 1 = 41$, $2n + 1 = 43$, and $2n + 3 = 45$.

6. A tangent line to the ellipse $x^2 + 4y^2 = 4$ meets the x-axis and y-axis at the points A and B, respectively. Find the minimum value of AB, the length of the line segment with endpoints A and B.

Solution. Using implicit differentiation, we have $y' = -x/(4y)$. Therefore, the line tangent to this ellipse at the point (x_0, y_0) is given by

$$y = -\frac{x_0}{4y_0}(x - x_0) + y_0.$$

Setting $x = 0$ gives

$$y = \frac{x_0^2}{4y_0} + y_0 = \frac{x_0^2 + 4y_0^2}{4y_0} = \frac{1}{y_0},$$

and setting $y = 0$ gives

$$x = \frac{4y_0^2}{x_0} + x_0 = \frac{x_0^2 + 4y_0^2}{x_0} = \frac{4}{x_0}.$$

(Recall that since (x_0, y_0) is on the ellipse, we have $x_0^2 + 4y_0^2 = 4$.) Therefore, $(AB)^2 = 16/x_0^2 + 1/y_0^2$. We wish to minimize $16/x^2 + 1/y^2$, for $-2 \le x \le 2$, subject to the constraint $x^2 + 4y^2 = 4$. This can be done with Lagrange multipliers or with straight substitution. Omitting the details, the minimum value of $(AB)^2$ occurs when $x^2 = 8/3$ and $y^2 = 1/3$. At these points, $(AB)^2 = 9$, so our answer is 3.

7. Find $\int_0^1 x \arcsin x \, dx$.

Solution. Let I denote the above integral. Using integration by parts with $u = \arcsin x$ and $dv = x\,dx$, we get

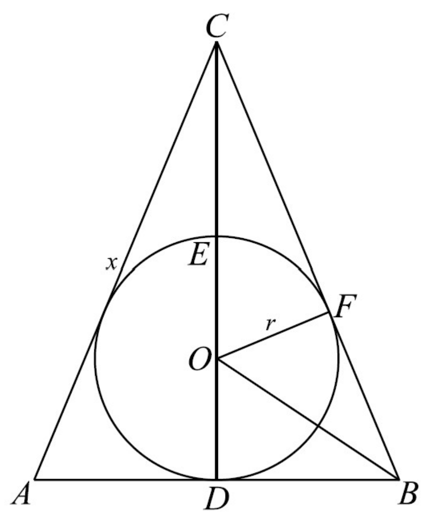
$$I = \frac{x^2}{2} \arcsin x \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{\sqrt{1-x^2}} \, dx.$$

Evaluating the first term and using the substitution $\sin\theta = x$ in the second term followed by the trig identity $\sin^2\theta = (1 - \cos(2\theta))/2$, gives

$$\begin{aligned} I &= \frac{\pi}{4} - \frac{1}{2} \int_0^{\pi/2} \sin^2\theta \, d\theta \\ &= \frac{\pi}{4} - \frac{1}{4} \int_0^{\pi/2} (1 - \cos(2\theta)) \, d\theta \\ &= \frac{\pi}{4} - \frac{1}{4} \left(\theta - \frac{1}{2} \sin(2\theta) \right) \Big|_0^{\pi/2} \\ &= \frac{\pi}{4} - \frac{\pi}{8} \\ &= \frac{\pi}{8}. \end{aligned}$$

8. In a magical isosceles triangle $\triangle ABC$, we have $|AC| = |BC|$. Let D be the midpoint between A and B. The inscribed circle of $\triangle ABC$ intersects the line segment CD in a point E that is in the interior of the triangle. Suppose that $|AB| = 15$ and $|CE| = 8$. Determine $|AC|$.

Solution. Consider the following sketch of the triangle $\triangle ABC$ with inscribed circle.



We let $x = |AC|$, r be the radius of the inscribed circle, O be the center of the inscribed circle, and F be the intersection of the inscribed circle with the segment BC. We note that the right triangle ODB is congruent to the right triangle OFB. Since $|DB| = \dfrac{15}{2}$, it follows that $|FB| = \dfrac{15}{2}$. So, $|CF| = x - \dfrac{15}{2}$.

Now, the smaller right triangle OFC is similar to the larger right triangle BDC. Hence,

$$\frac{|\,OF\,|}{|\,DB\,|} = \frac{|\,CF\,|}{|\,CD\,|} = \frac{|\,CO\,|}{|\,BC\,|}$$

Substituting in our variables gives,

$$\frac{r}{15/2} = \frac{x - 15/2}{2r + 8} = \frac{r + 8}{x}$$

On cross-multiplying the first and second fractions above, we find that

$$\frac{15}{2}\left(x - \frac{15}{2}\right) = r(2r + 8)$$

On cross-multiplying the first and third fractions above, we find that

$$\begin{aligned} rx &= \frac{15}{2}(r + 8) \\ rx &= \frac{15}{2}r + 60 \\ r\left(x - \frac{15}{2}\right) &= 60 \\ x - \frac{15}{2} &= \frac{60}{r} \end{aligned}$$

Substituting this into the equation above gives,

$$\begin{aligned} \frac{15}{2}\left(x - \frac{15}{2}\right) &= r(2r + 8) \\ \frac{15}{2}\left(\frac{60}{r}\right) &= r(2r + 8) \\ \frac{450}{r} &= 2r^2 + 8r \\ 450 &= 2r^3 + 8r^2 \\ 2r^3 + 8r^2 - 450 &= 0 \\ r^3 + 4r^2 - 225 &= 0 \\ (r - 5)(r^2 + 9r + 45) &= 0 \end{aligned}$$

Using the quadratic forumla for the quadradic equation does not give real roots:

$$\begin{aligned} r &= \frac{-9 \pm \sqrt{9^2 - 4(1)(45)}}{2} \\ &= \frac{-9 \pm \sqrt{81 - 180}}{2} \\ &= \frac{-9 \pm \sqrt{-99}}{2} \end{aligned}$$

Thus the only real root to the original cubic equation is $r = 5$. Substituting this back into the equation above allows us to solve for $\mid AC \mid = x$:

$$\begin{aligned} x - \frac{15}{2} &= \frac{60}{r} \\ x - \frac{15}{2} &= \frac{60}{5} \\ x - \frac{15}{2} &= 12 \\ x &= \frac{39}{2} \end{aligned}$$

9. Determine the number of ways that one can tile a 4×13 rectangle with 4×1 rectangles.

Solution. Let T_n denote the number of tilings of a $4 \times n$ rectangle with 4×1 rectangles. The upper left corner of the $4 \times n$ rectangle is either covered with a vertical 4×1 rectangle, leaving a $4 \times n-1$ rectangle left to tile, or a horizontal 4×1 rectangle with 3 horizontal 4×1 rectangles underneath it, leaving a $4 \times n - 4$ rectangle left to tile. Thus, for $n \geq 5$, $T_n = T_{n-1} + T_{n-4}$. Noting $T_1 = 1, T_2 = 1, T_3 = 1$ and $T_4 = 2$, we use the recurrence relation to get $T_5 = 3, T_6 = 4, T_7 = 5, T_8 = 7, T_9 = 10, T_{10} = 14, T_{11} = 19, T_{12} = 26$ and finally $T_{13} = 36$.

10. Katie and David are playing a game in which Katie rolls a fair n-sided die and David rolls a fair m-sided die. Katie rolls first and the winner is the first person to roll a 1. Katie and David are equally likely to win the game. How are m and n related?

Solution. The probability that Katie wins is $\frac{1}{n} + \left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)\left(\frac{1}{n}\right) + \left(\frac{n-1}{n}\right)^2\left(\frac{m-1}{m}\right)^2\left(\frac{1}{n}\right) + \cdots$, an infinite geometric series which sums to $\frac{\frac{1}{n}}{1-\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)}$. This must equal $\frac{1}{2}$ and thus

$$\frac{1}{n} = \frac{1}{2} - \frac{1}{2}\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right)$$

$$\frac{1}{2}\left(\frac{n-1}{n}\right)\left(\frac{m-1}{m}\right) = \frac{n-2}{2n}$$

$$\frac{m-1}{m} = \frac{n-2}{n-1}$$

$$m = n - 1$$

2022 Leo Schneider Student Team Competition (with solutions)

1. Find (with proof) all whole number values of n so that the binary expansion of $n!$ contains exactly two 1's.

   Solution. This means that $n! = 2^a + 2^b$, for some non-negative integers $a \neq b$. Reducing both sides mod 7, we notice that 2^a is congruent to either 1, 2, or 4 mod 7. This means that $2^a + 2^b$ can never be congruent to 0 mod 7, and thus $n \leq 7$. Subsequently, n from 1 to 6 can easily be checked by hand. Only $3! = 4 + 2$ and $4! = 16 + 8$ satisfy the given condition.

2. Consider a game played on a finite sequence of positive integers in which two types of moves, A and B, are allowed: A move of type A ("Add") replaces two adjacent integers in the sequence by their sum; for example $(\dots, 20, 11, \dots) \xrightarrow{A} (\dots, 31, \dots)$. A move of type B ("Break up") replaces a multi-digit integer in the sequence by the sequence of its *nonzero* decimal digits; for example, $(\dots, 2021, \dots) \xrightarrow{B} (\dots, 2, 1, 1, \dots)$.
   The moves may be combined in any manner. For example, given the sequence $(3, 14, 159, 26)$, a possible sequence of moves is the following:

   $$(3, 14, 159, 26) \xrightarrow{A} (17, 159, 26) \xrightarrow{B} (17, 1, 5, 9, 26) \xrightarrow{A} (18, 5, 9, 26) \xrightarrow{A} (18, 5, 35)$$
   $$\xrightarrow{A} (18, 40) \xrightarrow{B} (18, 4) \xrightarrow{B} (1, 8, 4) \xrightarrow{A} (9, 4) \xrightarrow{A} (13) \xrightarrow{B} (1, 3) \xrightarrow{A} (4)$$

   Once the sequence is reduced to a single one-digit number, any further moves will leave it unchanged, the game terminates, and we call the final number obtained the *terminal number* of the game. Suppose this game is played on the sequence $(1, 2, 3, 4, \dots, 2022)$. What is the terminal number? (For full credit, you must also show that this number is unique.)

   Solution. The key observation is that the remainder modulo 9 of the sum of all terms of the sequence is an invariant under both moves. This is obvious in the case of a move of type A since such a move leaves the sum of all terms unchanged. In the case of a move of type B, this follows from the divisibility test for 9, according to which the sum of decimal digits of a positive integer is congruent to the integer itself.
   Since the terminal number is a single-digit positive integer, it is uniquely determined by the remainder modulo 9 of the sum of all terms of the sequence. For the given sequence $(1, 2, 3, \dots, 2022)$, we have

   $$\sum_{i=1}^{2022} i = \frac{2022 \cdot 2023}{2} = 1011 \cdot 2023 \equiv 3 \cdot 7 = 21 \equiv 3 \bmod 9,$$

   So the terminal number for this sequence is 3.

3. Let $f(x) = x^2 + bx + c$ where b and c are real numbers between -1 and 1, selected uniformly at random. What is the probability that both roots of $f(x)$ are positive real numbers?

   Solution. The roots of $f(x)$ are given by $\frac{-b \pm \sqrt{b^2 - 4c}}{2}$, and thus the roots are both positive real numbers if and only if $b^2 \geq 4c$, $b < 0$, and $c > 0$. Considering (b, c) to be coordinates in the Cartesian plane, the set of all possible values for (b, c) is represented by a 2×2 square with area 4. The set of coordinates for which both roots are positive real numbers is represented by the region in the upper left part of the square and under the curve $c = b^2/4$. This region has area $\int_{-1}^{0} \frac{b^2}{4} db = \frac{b^3}{12}\Big|_{-1}^{0} = \frac{1}{12}$. Thus, the probability both roots of $f(x)$ are positive real numbers is $\frac{\frac{1}{12}}{4} = \frac{1}{48}$.

4. Suppose that A and B are two $n \times n$ matrices. Prove that

$$(A + AB^{-1}A)^{-1} + (A + B)^{-1} = A^{-1},$$

assuming that all these inverses exist.

Solution. First, note that $(XY)^{-1} = Y^{-1}X^{-1}$. Now,

$$\begin{aligned}
(A + AB^{-1}A)^{-1} + (A + B)^{-1} &= [AB^{-1}(B + A)]^{-1} + (A + B)^{-1} \\
&= (B + A)^{-1}BA^{-1} + (A + B)^{-1} \\
&= (A + B)^{-1}BA^{-1} + (A + B)^{-1} \\
&= (A + B)^{-1}(BA^{-1} + I) \\
&= (A + B)^{-1}(B + A)A^{-1} \\
&= (A + B)^{-1}(A + B)A^{-1} \\
&= A^{-1}.
\end{aligned}$$

5. An urn contains 3 red balls and 2 blue balls. A second urn contains 2 red balls and an unknown number of blue balls. Two balls are drawn at random from the first urn and placed in the 2nd urn. A ball is then randomly drawn from the 2nd urn and the probability that it is blue is $\frac{4}{5}$. How many blue balls are initially in the 2nd urn?

Solution. Let B be the number of blue balls in the 2nd urn. The probability of drawing two red balls from the first urn is $\frac{\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}$ and then the probability of drawing a blue ball from the 2nd urn is $\frac{B}{B+4}$. The probability of drawing one red ball and one blue ball from the first urn is $\frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \frac{6}{10}$ and then the probability of drawing a blue ball from the 2nd urn is $\frac{B+1}{B+4}$. The probability of drawing two blue balls from the first urn is $\frac{\binom{2}{2}}{\binom{5}{2}} = \frac{1}{10}$ and then the probability of drawing a blue ball from the 2nd urn is $\frac{B+2}{B+4}$. Therefore,

$$\frac{3}{10} \times \frac{B}{B+4} + \frac{6}{10} \times \frac{B+1}{B+4} + \frac{1}{10} \times \frac{B+2}{B+4} = \frac{4}{5};$$

which means $3B + 6B + 6 + B + 2 = 8B + 32$, or $B = 12$.

6. Given an equilateral triangle $\triangle ABC$ with sides of length 1 unit and with an arbitrary interior point P, draw the line segments $\overline{\text{PD}}$, $\overline{\text{PE}}$ and $\overline{\text{PF}}$ that are perpendicular to the 3 sides of the triangle. Let $s = PD + PE + PF$, the sum of the lengths of the 3 segments. Find the minimal value of s.

Solution. For any interior point P, $s = \frac{\sqrt{3}}{2}$. Draw segments from each of the vertices of $\triangle ABC$ to the point P. These segments will partition the equilateral triangle into 3 smaller triangles and thus the area of the equilateral triangle is equal to the sum of the areas of these 3 smaller triangles. Thus, $\frac{\sqrt{3}}{4} = \frac{1}{2}PD \cdot 1 + \frac{1}{2}PE \cdot 1 + \frac{1}{2}PF \cdot 1 = \frac{1}{2}(PD + PE + PF)$. Hence, $s = \frac{\sqrt{3}}{2}$.

7. Find $\int \ln(x+\sqrt{x^2-1})\,dx$.

Solution #1. Here, we first notice that $\sqrt{x^2-1}$ suggests substituting $x = \sec(\theta)$, Then $dx = \sec(\theta)\tan(\theta)d\theta$. Then, since $\sqrt{x^2-1} = \sqrt{\sec(\theta)^2-1} = \sqrt{\tan(\theta)^2}$, this turns the integral into $\int \ln(\sec(\theta)+\tan(\theta))\cdot\sec(\theta)\tan(\theta)d\theta$. But now, $\ln(\sec(\theta)+\tan(\theta))$ is exactly the integral of $\sec(\theta)$. So we can solve this integral with integration by parts by setting $u = \ln(\sec(\theta)+\tan(\theta))$ and $dv = \sec(\theta)\tan(\theta)$, so that $v = \sec(\theta)$ and $du = \sec(\theta)$. Then the integration by parts formula gives us $\int \ln(\sec(\theta)+\tan(\theta))\cdot\sec(\theta)\tan(\theta)d\theta = \sec(\theta)\ln(\sec(\theta)+\tan(\theta)) - \int \sec(\theta)^2$, which simplifies to $\sec(\theta)\ln(\sec(\theta)+\tan(\theta)) - \tan(\theta)$. Converting back to x's using the fact that $\sec(\theta) = x$ and $\tan(\theta) = \sqrt{x^2-1}$, we get $x\ln(x+\sqrt{x^2-1}) - \sqrt{x^2-1} + C$.

Solution #2. We use integration by parts with $u = \ln(x+\sqrt{x^2-1})$ and $dv = dx$. Then $v = x$, and

$$du = \frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}\,dx = \frac{1}{\sqrt{x^2-1}}\,dx.$$

Therefore,

$$\begin{aligned}\int \ln(x+\sqrt{x^2-1})\,dx &= x\ln(x+\sqrt{x^2-1}) - \int \frac{x}{\sqrt{x^2-1}}\,dx \\ &= x\ln(x+\sqrt{x^2-1}) - \sqrt{x^2-1} + C.\end{aligned}$$

8. Show that 2022 divides $676^{10} - 1024$.

Solution. Let $n = 676^{10} - 1024$. Since $2022 = 2\cdot 3\cdot 337$, we need to show that 2, 3, and 337 divide n. Notice that $n = (2\cdot 338)^{10} - 2^{10} = 2^{10}(338^{10}-1)$, so it is clearly divisible by 2. Working mod 337, we have $338^{10} - 1 \equiv 1^{10} - 1 = 0$; and working mod 3, we have $338^{10} - 1 \equiv (-1)^{10} - 1 = 0$.

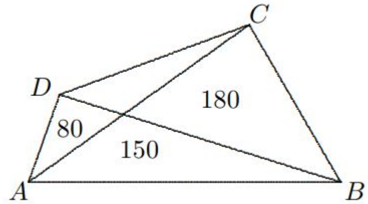
9. In polar coordinates, $r = 1+\cos\theta$ describes a cardioid. Find the maximum (vertical) distance between this cardioid and the x-axis.

Solution. We wish to maximize $y = r\sin\theta = (1+\cos\theta)\sin\theta$, for $0 \le \theta \le \pi$. Using the product rule and the fact that $\sin^2\theta = 1-\cos^2\theta$ gives us

$$y' = 2\cos^2\theta + \cos\theta - 1 = (2\cos\theta - 1)(\cos\theta + 1),$$

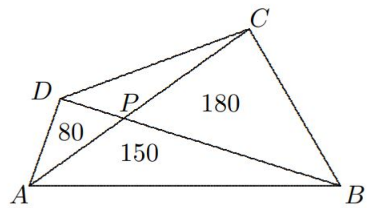
so the maximum value of y occurs when $\theta = \pi/3$. At this point, $y = 3\sqrt{3}/4$.

10. The quadrilateral $ABCD$ is partitioned into four triangles by means of the diagonals AC and BD. The areas of three of the triangles are indicated in the figure below.



What is the area of the quadrilateral?

Solution. Let P be the point of intersection of the diagonals, as shown below.



In general, let (XYZ) denote the area of triangle XYZ. Let h_D be the altitude from D to AC, and let h_B be the altitude from B to AC. Then,

$$\frac{(APD)}{(CPD)} = \frac{\frac{1}{2}(AP)h_D}{\frac{1}{2}(CP)h_D} = \frac{(AP)}{(CP)},$$

and,

$$\frac{(ABP)}{(CBP)} = \frac{\frac{1}{2}(AP)h_B}{\frac{1}{2}(CP)h_B} = \frac{(AP)}{(CP)}.$$

So,

$$\frac{80}{(CPD)} = \frac{(AP)}{(CP)} = \frac{150}{180}$$

Thus, $(CPD) = 96$, and the area of the quadrilateral is $180 + 150 + 80 + 96 = 506$.

~~2020~~ 2021 Leo Schneider Student Team Competition (with solutions)

1. Find the last two digits of $2021! + 2020! + 2019! + \cdots + 2! + 1!$.

   Solution. Working (mod 100), we need the last two digits of $1! + 2! + \cdots + 9! \equiv 1 + 2 + 6 + 24 + 20 + 20 + 40 + 20 + 80 = 213 \equiv 13$.

2. Find $\int_{-1}^{8} \sqrt{1 + \sqrt{1+x}}\, dx$.

   Solution. We use substitution with $u = \sqrt{1 + \sqrt{1+x}}$. Then $(u^2 - 1)^2 = 1 + x$ and so $2(u^2 - 1)2u\, du = dx$. Then

$$\begin{aligned} \int_{-1}^{8} \sqrt{1+\sqrt{1+x}}\, dx &= 4 \int_{1}^{2} (u^4 - u^2)\, du \\ &= 4 \left(\frac{u^5}{5} - \frac{u^3}{3} \right) \Bigg|_1^2 \\ &= \frac{232}{15} \end{aligned}$$

3. How many sequences of length 10 consisting of letters drawn with replacement from the set {M,A,T,H} have the letters in alphabetic order?

   Solution. This involves counting arrangements of indistinguishable letters into 4 distinct, some possibly empty, boxes (one box for each letter in the set). We can imagine this using the traditional "stars and bars" argument in combinatorics, by determining which of the $10+4-1 = 13$ positions (the 10 locations of the sequence plus 3 bars splitting up the letters) to place the 3 bars. Once the arrangement has been set, each bar represents the point where the sequence jumps to the next letter in the alphabet. (For example, $*****||*|****$ would represent the sequence AAAAAMTTTT.) Thus, there are $\binom{13}{3} = 286$ different sequences.

4. Find the sum of the digits of the number $9 + 99 + 999 + \cdots + 999\cdots9$, where the last number in our sum consists of 2021 9s.

   Solution. We can write the sum as

$$\begin{aligned} (10^1 - 1) + (10^2 - 1) + (10^3 - 1) + \cdots + (10^{2021} - 1) &= 10 + 10^2 + 10^3 + \cdots + 10^{2021} - 2021 \\ &= 111\cdots10 - 2021, \end{aligned}$$

where the first number in the last line has 2021 1s.

Since $111110 - 2021 = 109089$, our original number is $111\cdots109089$, where there are 2017 1s. The sum of this number's digits is therefore $2017 + 9 + 8 + 9 = 2043$.

5. The matrix $M = \begin{bmatrix} t & 1-t \\ 1 & 2t \end{bmatrix}$ has 1 as an eigenvalue. What are the possibilities for the other eigenvalue?

Solution. The characteristic polynomial of M is $(t-\lambda)(2t-\lambda)-(1-t) = 2t^2-3t\lambda+(t+\lambda^2-1)$. Since $\lambda = 1$ is a root of this polynomial, we know $2t^2 - 2t = 0$. This gives $t = 1$ or $t = 0$. Checking these possibilities give $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, which has eigenvalues 1 and 2, or $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which has eigenvalues 1 and -1. Thus the possible second eigenvalues are 2 or -1.

(Note: This problem can also be reasonably solved using the determinant and trace of M itself, or by assuming $M \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ and deducing a relationship between x and y.)

6. Suppose a and b are real numbers such that

$$\lim_{x\to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} = \frac{1}{2}.$$

Determine all possible ordered pairs (a, b).

Solution. Since $\sin x = 0$ when $x \to 0$, the given limit must be in indeterminate form, $\frac{0}{0}$. So, by L'Hôpital's Rule,

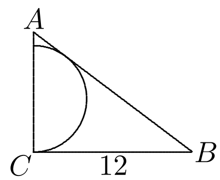
$$\lim_{x\to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} = \lim_{x\to 0} \frac{2\sin x \cos x}{ae^{ax} - b}.$$

Now, note that as $x \to 0$, $2\sin x \cos x \to 0$ but $ae^{ax} - b \to a - b$. So if $a \neq b$, the limit is 0, so we let $a = b$. Then, by L'Hôpital's Rule again, we get

$$\begin{aligned} \lim_{x\to 0} \frac{\sin^2 x}{e^{ax} - bx - 1} &= \lim_{x\to 0} \frac{-2\sin^2 x + 2\cos^2 x}{a^2 e^{ax}} \\ &= \lim_{x\to 0} \frac{2(\cos^2 x - \sin^2 x)}{a^2 e^{ax}} \\ &= \frac{2(1-0)}{a^2} \\ &= \frac{2}{a^2} \end{aligned}$$

Then, $\frac{2}{a^2} = \frac{1}{2}$ means $a^2 = 4$ so $a = \pm 2$. Thus, the only solutions are $(2, 2)$ and $(-2, -2)$.

7. Consider the following right triangle, with $m(BC) = 12$:



A semicircle is drawn with a diameter along side AC. One endpoint of the diameter is C, and the other is exactly one unit away from A. If the semicircle is tangent to side AB, find the radius of the circle.

Solution. Let O be the center of the circle, and let D be the point of tangency along AB. Let r denote the radius of the circle and y the length of AD. Then ΔAOD is a right triangle, with $m(AO) = r+1$ and $m(OD) = r$. Then by the Pythagorean theorem, $(r+1)^2 = y^2 + r^2$, or $y^2 = 2r + 1$. But also, ΔAOD is similar to ΔABC, so that $\frac{y}{r} = \frac{2r+1}{12}$. Solving for y and squaring gives $y^2 = \left(\frac{(2r+1)r}{12}\right)^2$. Setting this equal to $2r+1$, we can reduce to the cubic $2r^3 + r^2 - 144 = 0$. This has only one real root $r = 4$. (This means ΔABC has side lengths 9, 12, 15, and ΔAOD has side lengths 3,4,5.)

8. Let $f(x) = x\cos x$. Find, with proof, $f^{(2021)}(0)$, the 2021st derivative of f evaluated at 0.

Solution. The Taylor series of f, centered at 0, is given by

$$x(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots.$$

Since the coefficient of the x^{2021} term is $\frac{1}{2020!}$, we have that

$$f^{(2021)}(0) = 2021! \cdot \frac{1}{2020!} = 2021.$$

(Alternatively, one can show by induction that

$$f^{(2n+1)}(x) = (-1)^n(-x\sin x + (2n+1)\cos x),$$

for all $n \geq 0$, and then plug in $n = 1010$ and $x = 0$.)

9. Let $P(x)$ be a quadratic polynomial with real coefficients such that $P(3) = 2021$ and

$$P(x) = P(0) + P(1)x + P(2)x^2$$

for all real x. What is $P(-1)$?

Solution. Plugging $x = 1$ and $x = 2$ into the given equality yields the system of equations,

$$P(1) = P(0) + P(1) + P(2) \qquad \text{and} \qquad P(2) = P(0) + 2P(1) + 4P(2)$$

The first equality simplifies to $P(0) = -P(2)$ and plugging this into the second equation yields $P(1) = -P(2)$. So, we have the equality,

$$P(x) = -P(2) - P(2)x + P(2)x^2 = P(2)(x^2 - x - 1)$$

Now, since $P(3) = 2021$, we have,

$$2021 = P(2)(3^2 - 3 - 1) = 5P(2)$$

so $P(2) = \frac{2021}{5}$. Finally, $P(x) = \frac{2021}{5}(x^2 - x - 1)$, so $P(-1) = \frac{2021}{5}(1 + 1 - 1) = \frac{2021}{5}$.

10. You roll three fair six-sided dice. Find the probability that the product of the three numbers rolled is a perfect square.

Solution. Let N be the product of the three dice. The only possible square values of N are 1, 4, 9, 16, 25, 36, 64, 100, and 144. The only way to get $N = 1$ is to roll three ones, and the only way to get $N = 64$ is to roll three fours. There are three ways to get each of $N = 9$ (two threes and a one), $N = 25$ (two fives and a one), $N = 100$ (two fives and a four), and $N = 144$ (two sixes and a four). There are six ways to get both of $N = 4$ (either two ones and a four or two twos and a one) and $N = 16$ (two fours and a one or two twos and a four). Finally, there are twelve ways to get $N = 36$ (either two sixes and a one; two threes and a four; or one two, one three, and one six).

Therefore, in total, there are 38 ways (where order matters!) to have N be a perfect square, and our probability is $\frac{38}{216} = \frac{19}{108} \approx 0.176$.

2019 Leo Schneider Student Team Competition (with solutions)

1. For each $a < 0$, consider the line perpendicular to the curve $y = x^2$ at $x = a$. This line intersects the curve at another point. Find the minimum possible value of the x-coordinate of this second intersection point.

   Solution: Since the slope of $y = x^2$ at a is $2a$, the slope of the perpendicular at this point is $-1/(2a)$. The equation of the perpendicular is then

   $$y - a^2 = \frac{-1}{2a}(x - a).$$

   Therefore, the x-coordinate of the second intersection point satisfies

   $$x^2 - a^2 = \frac{-1}{2a}(x - a).$$

   Thus, $x = -a - \frac{1}{2a}$. Using calculus, we see that this function attains its minimum value of $\sqrt{2}$ when $a = -1/\sqrt{2}$.

2. Compute $\displaystyle\int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx$.

   Solution: We make the rationalizing substitution $t = \sqrt[6]{x}$, so that $x = t^6$ and $dx = 6t^5 dt$. Then the integral becomes $\int \frac{6t^5}{t^3+t^2} dt = \int \frac{6t^3}{t+1}$. From long division, this is equivalent to $\int 6t^2 - 6t + 6 - \frac{6}{t+1} dt$. Finally, we can integrate directly to get $2t^3 - 3t^2 + 6t - 6\ln|t+1|$. Substituting back in for t, we get $2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln|\sqrt[6]{x} + 1| + C$.

3. Find the value of $\displaystyle\int_0^\infty \left(\frac{3}{4}\right)^{\lfloor x \rfloor} dx$, where $\lfloor x \rfloor$ is the **floor function**, which gives the largest integer less than or equal to x.

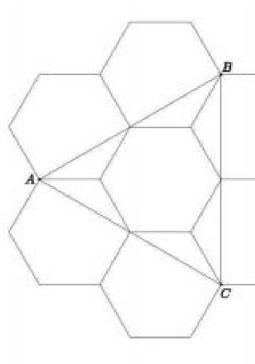
   Solution: Note that $f(x) = \left(\frac{3}{4}\right)^{\lfloor x \rfloor}$ is a step function with value $(3/4)^0$ on [0, 1), value $(3/4)^1$ on [1, 2), value $(3/4)^2$ on [2, 3), ... Thus we can write the integral as a Riemann sum as follows:

   $$\int_0^\infty \left(\frac{3}{4}\right)^{\lfloor x \rfloor} dx = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1 - \frac{3}{4}} = 4.$$

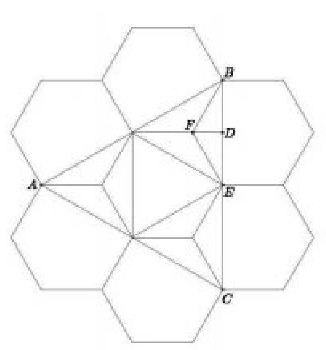
4. Three people, named A, B and C, throw a die alternately. First A throws, then B, then C, and this keeps repeating. What is the probability that A throws the first six, B the second six and C the third six?

   Solution: To find the exact probability, the probability that A throws a 6 on the first try is 1/6. Their probability of rolling the first 6 on the second try is $(5/6)^3 * 1/6$. (This assumes that on the first time through, no one throws a 6, which has probability $(5/6)^3$, but then A does throw a 6 on the second time through.) Continuing in this way, the total probability that A throws the first 6 is $1/6 + 1/6(5/6)^3 + (1/6)(5/6)^6 + \ldots$. This is a geometric series with ratio $(5/6)^3$, so it converges to 36/91. Then if A throws the first 6, the process starts over but now B is going first. So their probability of rolling the next 6 is also 36/91. The same reasoning applies to C if B throws the second 6. So the total probability that the first three 6's come up for A-B-C is $(36/91)^3$.

5. Six regular hexagons surround a regular hexagon of side length 1 as shown. What is the area of $\triangle ABC$?



Solution: Label points E and F as shown in the figure, and let D be the midpoint of $\bar{BE}$. Because $\triangle BFD$ is a 30-60-90° triangle with hypotenuse 1, then length of $\bar{BD}$ is $\frac{\sqrt{3}}{2}$. Therefore, the length of $\bar{BC}$ (4 times the length of $\bar{BD}$) is $2\sqrt{3}$. The formula for the area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. Thus, the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4}(2\sqrt{3})^2 = 3\sqrt{3}$.



6. Find $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{\sqrt{n^2+k}}$.

Solution: For $0 \le k \le n$ we have $\sqrt{n^2+n} \ge \sqrt{n^2+k} \ge \sqrt{n^2}$, or

$$\frac{1}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+k}} \le \frac{1}{n}.$$

Therefore

$$\frac{n+1}{\sqrt{n^2+n}} \le \sum_{k=0}^{n}\frac{1}{\sqrt{n^2+k}} \le \frac{n+1}{n};$$

and, by the Squeeze Theorem, $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1}{\sqrt{n^2+k}} = 1$.

7. A two-digit number is uniquely determined by the answers to the following yes/no questions.

   1. Is the number divisible by 2?
   2. Is the number divisible by 3?
   3. Is the number divisible by 5?
   4. Is the number divisible by 7?

   If the answer to the first question is “yes”, what is the number?

   Solution: Looking at all 16 possible combinations of yes/no answers, we see that only two of them (NNYY and YNYY) yield a unique two-digit number. YNYY yields the number 70.

8. Write 2019 as the difference of squares of positive integers in two different ways.

Solution: Suppose $2019 = a^2 - b^2 = (a-b)(a+b)$, for some positive integers a and b. Since $2019 = 1(2019) = 3(673)$, we have either $a+b = 2019$ and $a-b = 1$, or $a+b = 673$ and $a-b = 3$. In the first case we get $a = 1010$ and $b = 1009$, and in the second case we get $a = 338$ and $b = 335$. Therefore

$$2019 = 1010^2 - 1009^2 = 338^2 - 335^2.$$

9. What is the largest integer a such that 7^a divides 1000! ?

Solution: Since $994/7 = 142$, there are 142 multiples of 7 in the product 1000!. However, this includes the 20 numbers $49, 98, \ldots 980$ which contain an additional factor of 7, and then 343 and 686, which contain yet another factor of 7. Thus, the total number of factors of 7 in 1000! is $142 + 20 + 2 = 164$.

10. The base 5 representation of a positive integer has 3 digits. Reversing those digits gives the base 7 representation of the same number. What are the possible base 10 representations of that number?

Solution: The number can be expressed as

$$a \cdot 5^0 + b \cdot 5^1 + c \cdot 5^2 = c \cdot 7^0 + b \cdot 7^1 + a \cdot 7^2$$

which reduces to $48a + 2b = 24c$, or $24a + b = 12c$. Since a, b, and c, must be integers between 0 and 4, we can rule out the cases where c is 0, 1, and 3, as they lead to values of b that are outside of our range. So c must be 2 or 4. If $c = 2$, then $b = 0$ and $a = 1$, which means the number is $1 + 0 \cdot 5 + 2 \cdot 5^2 = 51$. If $c = 4$, then $b = 0$ and $a = 2$, which means the number is $2 + 0 \cdot 5 + 4 \cdot 5^2 = 102$. Thus the possible base 10 representations are 51 and 102.

2018 Leo Schneider Student Team Competition (SOLUTIONS)

1. Find $\displaystyle\int \frac{x^5}{(x^3+1)^2}\,dx$.

   Solution. We can substitute $u = x^3+1$, so that $du = 3x^2\,dx$, and $u-1 = x^3$. This means $x^5 = \frac{1}{3}(u-1)\,du$. Then the integral becomes $\displaystyle\frac{1}{3}\int\frac{u-1}{u^2} = \frac{1}{3}\int \frac{1}{u} - \frac{1}{u^2}du = \frac{1}{3}\left(\ln(|x^3+1|) + \frac{1}{x^3+1}\right) + C$.

2. Evaluate the following limit:
$$\lim_{x\to 0}\frac{\sin\arctan x - \tan\arcsin x}{\arcsin\tan x - \arctan\sin x}.$$

   Solution. Computing the limit is possible if we remember (or calculate) the Taylor series for the functions:
$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \text{terms of order 5 or higher}\\ \tan x &= x + \frac{x^3}{3!} + \text{terms of order 5 or higher}\\ \arcsin x &= x + \frac{x^3}{3!} + \text{terms of order 5 or higher}\\ \arctan x &= x - \frac{x^3}{3!} + \text{terms of order 5 or higher.}\end{aligned}$$

   After composition and some manipulation, we get
$$\lim_{h\to 0}\frac{-x^3 + \text{terms of order 5 or higher}}{x^3 + \text{terms of order 5 or higher}} = -1.$$

3. Let $\left(1+\sqrt{2}\right)^n = A_n + B_n\sqrt{2}$ with A_n and B_n rational numbers.

   (a) Express $\left(1-\sqrt{2}\right)^n$ in terms of A_n and B_n.

   (b) Compute $\displaystyle\lim_{n\to\infty}\frac{A_n}{B_n}$.

   Solution.
   (a) Prove by induction that $(1-\sqrt{2})^n = A_n - B_n\sqrt{2}$.
   (b) Since $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$ and $(1-\sqrt{2})^n = A_n - B_n\sqrt{2}$, solve to find:
$$A_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2},\quad B_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

   So,
$$\frac{A_n}{B_n} = \sqrt{2}\cdot\frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{(1+\sqrt{2})^n - (1-\sqrt{2})^n} = \sqrt{2}\cdot\frac{1+x^n}{1-x^n}$$
   where $x = \dfrac{1-\sqrt{2}}{1+\sqrt{2}}$. Since $|x|<1$, $x^n\to 0$ as $n\to\infty$. So $\displaystyle\lim_{n\to\infty}\frac{A_n}{B_n} = \sqrt{2}$.

4. You roll three fair dice. Find the probability that some subset of the numbers you rolled sums to 3. (This includes possible rolls such as 1, 5, 3.)

   Solution. We will count the number of combinations, out of the 6^3 total combinations, that "win". $6^3-5^3 = 91$ contain at least one 3. Of those remaining, if the first roll is a 4, 5, or 6, then the next two rolls must be a 1 and a 2 (in either order). This gives us 6 more winning rolls. If a 2 is rolled followed by a 1, then the third roll can be anything (other than a 3, which was already counted). If a

2 is rolled followed by anything other than a 1 or a 3, then the third roll must be a 1. This gives us 9 more winning rolls. If a 1 is rolled followed by a 2, then the third roll can be anything (other than a 3, which was already counted). If a 1 is rolled followed by a 4, 5, or 6, then the third roll must be a 2. This gives us 8 more winning rolls. This leaves 1,1,2 and 1,1,1 as the last two winning rolls. The answer is therefore

$$\frac{91+6+9+8+2}{6^3} = \frac{116}{216} = \frac{29}{54}.$$

5. What is the probability of an odd number of sixes turning up in a random toss of n fair dice?

Solution. For $0 \le k \le n$, the probability of k sixes turning up in a random toss of n fair dice is

$$\binom{n}{k}\left(\frac{5}{6}\right)^{n-k}\left(\frac{1}{6}\right)^{k};$$

hence, with $a = 5/6$ and $b = 1/6$, the required probability is

$$\begin{aligned} P &= \binom{n}{1}a^{n-1}b + \binom{n}{3}a^{n-3}b^3 + \binom{n}{5}a^{n-5}b^5 + \cdots \\ &= \text{ sum of the even-ranked terms in the expansion of } (a+b)^n \\ &= \frac{1}{2}\{(a+b)^n - (a-b)^n\} \\ &= \frac{1}{2}\left\{1 - \left(\frac{2}{3}\right)^n\right\}. \end{aligned}$$

6. $\triangle ABC$ satisfies $AC = 1$ and $\angle ACB$ is right. Points D and E are on $\overline{AB}$ with $\overline{CD}$ and $\overline{CE}$ trisecting $\angle ACB$. Suppose, moreover, that AB is the smallest integer such that AD, DE, and EB are distinct rational numbers. Find AB.

Solution. Without loss of generality, let D be between A and E. Put $k = AB$ and $\theta = \angle BAC$. Then $\sin\theta = \frac{\sqrt{k^2-1}}{k}$ and $\cos\theta = \frac{1}{k}$. Using the Law of Sines on $\triangle ACD$ gives us

$$\frac{AD}{1/2} = \frac{1}{\sin(150^\circ - \theta)} = \frac{1}{\frac{1}{2}\cdot\frac{1}{k} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{k^2-1}}{k}},$$

or $AD = \frac{k}{1+\sqrt{3}\sqrt{k^2-1}}$. Similarly, using the Law of Sines on $\triangle BEC$, we get $EB = \frac{k\sqrt{k^2-1}}{\sqrt{k^2-1}+\sqrt{3}}$. Now, $k = 2$ gives us $AD = \frac{1}{2}$ and $EB = 1$. But this means that $DE = \frac{1}{2}$, so the distinctness property is violated. The next value of k that "works" is $k = 7$, which gives us $AD = \frac{7}{13}$, $EB = \frac{28}{5}$, and $DE = \frac{56}{65}$. So $AB = 7$.

7. Let $A = \begin{bmatrix} -1 & 2018 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Compute $A^{2018}\mathbf{u}$.

Solution. We first compute $A\mathbf{u}$:

$$A\mathbf{u} = \begin{bmatrix} -1 & 2018 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\mathbf{u}$$

Thus, $A\mathbf{u} = -\mathbf{u}$. Repeatedly using this results, we have:

$$A^2\mathbf{u} = A(A\mathbf{u}) = A(-\mathbf{u}) = -A\mathbf{u} = -(-\mathbf{u}) = \mathbf{u}$$

$$A^3\mathbf{u} = A(A^2\mathbf{u}) = A(\mathbf{u}) = -\mathbf{u}$$

$$A^4\mathbf{u} = A(A^3\mathbf{u}) = A(-\mathbf{u}) = -A\mathbf{u} = -(-\mathbf{u}) = \mathbf{u}$$

So, $A^n\mathbf{u} = \begin{cases} -\mathbf{u} & \text{if } n \text{ is odd} \\ \mathbf{u} & \text{if } n \text{ is even} \end{cases}$

Therefore, $A^{2018}\mathbf{u} = \mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

8. Find all integers c such that $x^2 + c$ divides $x^3 + 4x^2 - x - 4$. Explain why you have found all the answers!

   Solution. If $x^2 + c$ divides $x^3 + 4x^2 - x - 4$, then the quotient must be a linear polynomial with leading coefficient 1, so $x^3 + 4x^2 - x - 4 = (x^2 + c)(x + a)$. Then we must have $ac = -4$ and $c = -1$, so that $a = 4$. This factorization does work, so $c = -1$ is the only solution.

9. Given that $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x, find the following integral:

$$\int_0^2 \lfloor x \rfloor - 2\left\lfloor \frac{x}{2} \right\rfloor \, dx.$$

   Solution. Since $\lfloor x \rfloor$ is 0 for $0 \le x < 1$ and is 1 for $1 \le x < 2$, and similarly $\left\lfloor \frac{x}{2} \right\rfloor$ is 0 for $0 \le x < 2$, the function we are integrating looks like a step function which is 0 for $0 \le x < 1$ and 1 for $1 \le x < 2$. Thus the integral is:

$$\int_0^2 \lfloor x \rfloor - 2\left\lfloor \frac{x}{2} \right\rfloor \, dx = \int_0^1 0 \, dx + \int_1^2 1 \, dx = 1.$$

10. Integers x, y, and z satisfy $x + 2y + 3z = 60$, $4x + 5y = 60$, and $z \ge 0$. What is the maximum value of the product xyz?

    Solution. Solving the given system for x and y gives us $x = 5z - 60$ and $y = 60 - 4z$. Then $xyz = -z(5z - 60)(4z - 60)$. Since $z \ge 0$, the maximum value of this cubic occurs between its two largest roots: $z = 12$ and $z = 15$. When $z = 13$, $x = 5$ and $y = 8$, giving us $xyz = 520$. When $z = 14$, $x = 10$ and $y = 4$, giving us $xyz = 560$. The maximum value of xyz is 560.

2017 Leo Schneider Student Team Competition (SOLUTIONS)

1. Find
$$\int_4^\infty \frac{1}{\sqrt{x}(x-1)}\,dx.$$

Solution. The substitution $u^2 = x$ transforms the integral into
$$\int_2^\infty \frac{2}{u^2-1}\,du.$$
We can rewrite this definite integral using partial fraction decomposition to get
$$\begin{aligned}\int_2^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du &= \ln \frac{u-1}{u+1}\bigg|_2^\infty \\ &= \ln 1 - \ln \frac{1}{3} \\ &= \ln 3.\end{aligned}$$

2. If
$$u = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots, v = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \cdots, w = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \cdots,$$
prove that
$$u^3 + v^3 + w^3 - 3uvw = 1.$$

Solution. The power series for u, v, and w converge for all x, and
$$\frac{du}{dx} = w, \quad \frac{dv}{dx} = u, \quad \frac{dw}{dx} = v,$$
as we see by differentiating them. Letting
$$f = u^3 + v^3 + w^3 - 3uvw,$$
we have
$$\begin{aligned} f' &= 3u^2u' + 3v^2v' + 3w^2w' - 3uvw' - 3uv'w - 3u'vw \\ &= 3u^2w + 3v^2u + 3w^2v - 3uv^2 - 3u^2w - 3vw^2 = 0.\end{aligned}$$
Thus $f =$ constant. But $f(0) = (u(0))^3 = 1$, so $f(x) = 1$ for all x.

3. Find $\lim_{n\to\infty} \frac{\binom{2(n+1)}{n}}{\binom{2n}{n+1}}$.

Solution.
$$\begin{aligned}\lim_{n\to\infty} \frac{\binom{2(n+1)}{n}}{\binom{2n}{n+1}} &= \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)!(n+1)!} \cdot \frac{(n+1)!(n-1)!}{(2n!)} \\ &= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{n(n+2)} = 4.\end{aligned}$$

4. Jason has an 8 a.m. class this semester, and he is not a morning person. If he wakes up before 7:30, he showers in the morning 60% of the time, but if he wakes up after 7:30, he showers only 10% of the time. He only manages to wake up before 7:30 25% of the time, but when he does, he makes it to class on time 80% of the time. When he doesn't wake up before 7:30, he only makes it to class on time 30% of the time. Assuming that on a given morning he takes a shower, what is the probability that he makes it to class on time?

Note. Thanks to Professor Larry Robinson at ONU for pointing out a flaw in the problem. Not enough information is given to find the solution. Can you find the flaw in the following solution?

Solution. Let A be the event that he wakes up on time, let B be the event that he showers in the morning, and let C be the event that he makes it to class on time. We are given $P(A) = .25$, $P(B|A) = .6$, $P(B|A') = .1$, $P(C|A) = .8$, and $P(C|A') = .3$. Then, we use Bayes' theorem to compute

$$P(A|B) = \frac{P(A \cap B)}{P(A \cap B) + P(A' \cap B)} = \frac{.15}{.15 + .075} = 2/3.$$

and

$$P(A'|B) = 1/3.$$

Then, we compute $P(C|B) = P(C \cap A|B) + P(C \cap A'|B) = P(A|B)P(C|A) + P(A'|B)P(C|A')$, which is $2/3 \cdot .8 + 1/3 \cdot .3 = .15\overline{3}$.

5. Two points are chosen at random (with a uniform distribution) from the unit interval $[0, 1]$. What is the probability that the points will be within a distance of $1/8$ of each other?

Solution. The probability is the area of the region defined by $0 \le x \le 1$, $0 \le y \le 1$, and $|x - y| \le 1/8$ (divided by the area of the unit square). But this is $1 - 2 \cdot (1 - (1/8))^2/2 = 2(1/8) - (1/8)^2$.

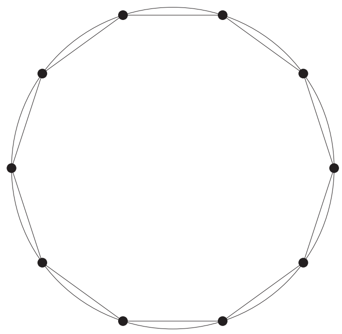
6. Michael Phelps' favorite hobby in his offtime is alchemy. He has a machine that can change 1 gold medal into 2 silver and 3 bronze, another machine that can turn 1 silver medal into 2 gold and 1 bronze, and a third machine that can turn 2 bronze medals into 3 gold and 1 silver. Starting with the 23 gold medals, 3 silver medals, and 2 bronze medals that he already has, can he use his machines to end up with a total of exactly 27 gold medals, 17 silvers, and 18 bronzes? Find a sequence of uses of the machines that accomplishes this or explain why it can't be done.

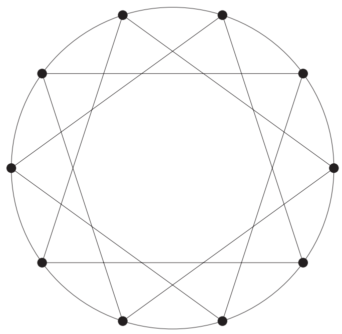
Solution. A series of transformations using these machines can be described as a linear combination of the vectors $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$. (The first entry in each vector describes the change in the number of gold medals, the second describes silvers, and the third bronze.) Michael wants to add 4 gold medals, 14 silvers, and 16 bronzes to his collection. Thus we want a solution of

$$\begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ 16 \end{bmatrix}.$$

However, solving this system gives the unique solution $x = 13/2, y = 3/2, z = 5/2$. So this series of transformations is not achievable since fractional numbers of uses of the machines are impossible.

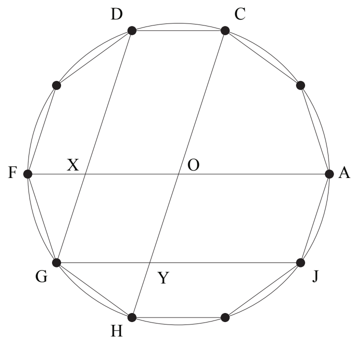
7. Ten equally spaced points are marked around the circumference of a circle. Consider connecting them into decagons two ways: first, by connecting all the nearest pairs of points, and second, by connecting every third point.





Prove that the difference in side lengths of these two decagons is equal to the radius of the circle.

Solution. Geometric solution (draw just the right lines and it falls out via parallelograms!)



8. Find the largest integer that divides $p^2 - 1$ for all primes $p > 3$.

Solution. If $p = 5$, then $p^2 - 1 = 24$, so the answer must be a divisor of 24. Since $p^2 - 1 = (p+1)(p-1)$, and $p > 3$ is odd, one of $(p+1)$ and $(p-1)$ will be divisible by 2 while the other will be divisible by 4. So the answer will be either 8 or 24. Furthermore, since $p > 3$ cannot be divisible by 3, either $(p+1)$ or $(p-1)$ must be divisible by 3. Thus the answer is 24.

9. What is the smallest number divisible by 225 that (written in base 10) contains only the digits 1 and 0? Explain your answer.

Solution. Since $225 = 25 \cdot 9$, the number in question must be divisible by 25, so that its final two digits are 0's, and the sum of its digits must be divisible by 9. The smallest such number is 11,111,111,100.

10. A sequence begins with a_1, a_2, and for $n > 2$ is defined by $a_n = a_{n-1} - a_{n-2}$. Find the sum of the first 2016 terms and defend your answer.

Solution. From the recursion we can write out some terms,

$$a_1, a_2, a_2 - a_1, -a_1, -a_2, a_1 - a_2, a_1, a_2, ...$$

to see that the sequence of terms will be periodic. Now let's look at S_n, the sum of the first n terms. Writing out some terms

$$a_1, a_1 + a_2, 2a_2, 2a_2 - a_1, a_2 - a_1, 0, a_1, a_2, ...$$

we see that it is periodic with $S_{6k} = 0$ for all k. Since $2016 = 0 \pmod 6$, we have that $S_{2016} = 0$.

2016 Leo Schneider Student Team Competition (SOLUTIONS)

1. Neither *Mathematica* nor *Maple* can find the exact value of the following definite integral. Can you? We think you can. Do it!

$$\int_0^2 (3x^2 - 3x + 1)\cos(x^3 - 3x^2 + 4x - 2)\,dx.$$

Solution: We first split up the integral as

$$\int_0^2 (3x^2 - 6x + 4)\cos(x^3 - 3x^2 + 4x - 2)\,dx + \int_0^2 (3x - 3)\cos(x^3 - 3x^2 + 4x - 2)\,dx,$$

If we set $u = x^3 - 3x^2 + 4x - 2$ in the first integral, we get $\int_{-2}^{2} \cos u\,du$ which equals $2\sin(2)$.

If we set $v = x - 1$ in the second integral, we get $\int_{-1}^{1} 3v\cos(v^3 + v)\,dv$. Applying the definition of even and odd function shows $\cos(v^3 + v)$ to be an even function, which means that $v\cos(v^3 + v)$ is an odd function and the second integral is 0.

Therefore, the original integral is equal to $2\sin(2)$.

2. In a certain group of cancer patients, each patient's cancer is classified in exactly one of the following five stages: stage 0, stage 1, stage 2, stage 3, or stage 4.

   (a) 75% of the patients in the group have stage 2 or lower

   (b) 80% of the patients in the group have stage 1 or higher

   (c) 80% of the patients in the group have stage 0,1, 3, or 4.

One patients is randomly selected. Calculate the probability that the selected patient's cancer is stage 1.

Solution: Let p_i be the probability that a randomly chosen patient's cancer is in stage i for $i = 0, 1, 2, 3, 4$. Combining the given information and the fact that the probabilities must sum to 1, we have that

$$p_0 + p_1 + p_2 + p_3 + p_4 = 1$$

$$p_0 + p_1 + p_2 = 0.75$$

$$p_1 + p_2 + p_3 + p_4 = 0.80$$

$$p_0 + p_1 + p_3 + p_4 = 0.80$$

Therefore, we know that

$$p_0 = (p_0 + p_1 + p_2 + p_3 + p_4) - (p_1 + p_2 + p_3 + p_4) = 1 - 0.8 = 0.2$$

and that

$$p_2 = (p_0 + p_1 + p_2 + p_3 + p_4) - (p_0 + p_1 + p_3 + p_4) = 1 - 0.8 = 0.2$$

from which we can conclude that

$$p_1 = 0.75 - p_0 - p_2 = 0.75 - 0.2 - 0.2 = 0.35.$$

3. Compute

$$\int_0^{\sqrt{\pi/3}} \sin x^2 \ dx + \int_{-\sqrt{\pi/3}}^{\sqrt{\pi/3}} x^2 \cos x^2 \ dx.$$

Solution: We have

$$\begin{aligned}\int_0^{\sqrt{\pi/3}} \sin x^2 \ dx + \int_{-\sqrt{\pi/3}}^{\sqrt{\pi/3}} x^2 \cos x^2 \ dx &= \int_0^{\sqrt{\pi/3}} \sin x^2 \ dx + 2\int_0^{\sqrt{\pi/3}} x^2 \cos x^2 \ dx \\ &= \int_0^{\sqrt{\pi/3}} [\sin x^2 + x\cos x^2 \cdot 2x] \ dx = \int_0^{\sqrt{\pi/3}} \frac{d}{dx}(x \sin x^2) \ dx \\ &= x \sin x^2 |_0^{\sqrt{\pi/3}} = \sqrt{\pi}/2.\end{aligned}$$

4. Let $a_1,\ a_2,\ a_3, \cdots$ be the increasing sequence of positive integers that are divisible by 2 or 5. The sequence begins 2, 4, 5, 6, 8, 10, 12, 14, 15, 16, ... Compute the sum of the following series.

$$\sum_{n=1}^{\infty} \frac{1}{2^{a_n}} = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \cdots$$

Solution: By inclusion-exclusion,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{2^{a_n}} &= \sum_{k=1}^{\infty} \frac{1}{2^{2k}} + \sum_{k=1}^{\infty} \frac{1}{2^{5k}} - \sum_{k=1}^{\infty} \frac{1}{2^{10k}} \\ &= \frac{1/4}{1-1/4} + \frac{1/32}{1-1/32} - \frac{1/1024}{1-1/1024} \\ &= \frac{1}{3} + \frac{1}{31} - \frac{1}{1023} \\ &= \frac{373}{1023}.\end{aligned}$$

5. Find all integer solutions (x, y) to the equation $xy = 5x + 11y$.

Solution:

First, if we re-write the equation $xy = 5x + 11y$ as $xy - 5x - 11y = 0$, we can add 55 to both sides to get $xy - 5x - 11y + 55 = 55$. So we have

$$(x - 11)(y - 5) = 55.$$

Now, x and y are supposed to be integers, so $x - 11$ and $y - 5$ should also be integers. Thus, the question is, how many different pairs of integers can we find whose product is 55? There are eight pairs:

$$(55, 1),\ (-55, -1),\ (1, 55),\ (-1, -55),\ (5, 11),\ (-5, -11),\ (11, 5),\ (-11, -5).$$

If $x - 5$ is to represent the first number and $y - 11$ is to represent the second number, then solving for x and y in each pair yields the eight possible solutions

$$x = 66,\ y = 6 \qquad x = -44,\ y = 4 \qquad x = 12,\ y = 60$$

$$x = 10,\ y = -50 \qquad x = 16,\ y = 16 \qquad x = 6,\ y = -6$$

$$x = 22,\ y = 10 \qquad x = 0,\ y = 0.$$

6. Find

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{2n}}\right).$$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{2n}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\cdots+\sqrt{\frac{n}{n+n}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+\cdots+\frac{1}{\sqrt{1+1}}\right)\\
&=\int_0^1\frac{1}{\sqrt{1+x}}\,dx=2\sqrt{1+x}\Big|_0^1=2(\sqrt{2}-1).
\end{aligned}$$

Alternate Solution: Using the Stolz-Cesàro Theorem (essentially the discrete case of L'Hôpital's Rule), rationalizing the denominator, and looking at the highest powers of n, we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{1}{\sqrt{n+k}}}{\sqrt{n}} &= \lim_{n\to\infty}\frac{\frac{1}{\sqrt{2n+1}}+\frac{1}{\sqrt{2n+2}}-\frac{1}{\sqrt{n+1}}}{\sqrt{n+1}-\sqrt{n}}\\
&= \lim_{n\to\infty}(\sqrt{n+1}+\sqrt{n})\left(\frac{1}{\sqrt{2n+1}}+\frac{1}{\sqrt{2n+2}}-\frac{1}{\sqrt{n+1}}\right)\\
&= \lim_{n\to\infty}\left(\frac{2\sqrt{n}}{\sqrt{2n}}+\frac{2\sqrt{n}}{\sqrt{2n}}-\frac{2\sqrt{n}}{\sqrt{n}}\right)\\
&= 2\sqrt{2}-2.
\end{aligned}$$

7. Let

$$f(r)=\sum_{j=2}^{2016}\frac{1}{j^r}=\frac{1}{2^r}+\frac{1}{3^r}+\cdots+\frac{1}{2016^r}.$$

Find

$$\sum_{k=2}^{\infty}f(k).$$

Solution:
After switching the order of summation, we note that the inner sum is a convergent geometric series. Thus, we can proceed as follows:

$$\begin{aligned}
\sum_{k=2}^{\infty}\sum_{j=2}^{2016}\frac{1}{j^k}&=\sum_{j=2}^{2016}\sum_{k=2}^{\infty}\frac{1}{j^k}=\sum_{j=2}^{2016}\frac{\frac{1}{j^2}}{1-\frac{1}{j}}\\
&=\sum_{j=2}^{2016}\frac{1}{j^2-j}=\sum_{j=2}^{2016}\frac{1}{j(j-1)}=\sum_{j=2}^{2016}\left(\frac{1}{j-1}-\frac{1}{j}\right)\\
&=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{2015}-\frac{1}{2016}=\frac{2015}{2016}.
\end{aligned}$$

8. Urn 1 contains 10 balls: 4 red and 6 blue. A second urn originally contains 15 red balls and an unknown number of blue balls. Joe draws a single ball from the first urn and places it into the second urn. He then draws a ball from the second urn. Prior to conducting the experiment, the probability that both balls drawn will be of the same color is 0.5 What is the probability the ball drawn from urn 1 was red given that the ball drawn from urn 2 was red?

Solution: Let x be the number of blue balls in the second urn. The probability of drawing the same color ball from both urns is

$$\text{Pr(drawing both red)} + \text{Pr(drawing both blue)} = \left(\frac{4}{10}\right)\left(\frac{16}{x+16}\right) + \left(\frac{6}{10}\right)\left(\frac{x+1}{x+16}\right)$$

Busting out some amazing algebra, we set this to 0.5 and solve for x.

$$\begin{aligned}\left(\frac{4}{10}\right)\left(\frac{16}{x+16}\right) + \left(\frac{6}{10}\right)\left(\frac{x+1}{x+16}\right) &= 0.5\\ \frac{64+6x+6}{10x+160} &= 0.5\\ 6x+70 &= 5x+80\\ x &= 10\end{aligned}$$

Next, P(Red Draw 1 | Red Draw 2) = P(Red Draw 1 AND Red Draw 2) / P(Red Draw 2)

P(Red Draw 1 AND Red Draw 2) = (4/10) (16/26) = 64/260.

P(Red Draw 2) = P(Red Draw 1 AND Red Draw 2) + P(Blue Draw 1 AND Red Draw 2) = 16/65 + (6/10)(15/26) = 16/65 + 90/260 = 64/260 + 90/260 = 154/260 = 77/130.

P(Red Draw 1 | Red Draw 2) = (64/260) / (154/260) = 64/154 = 32/77.

9. The rhombicosidodecahedron is an Archimedean solid with faces that are equilateral triangles, squares, and regular pentagons. It has 60 vertices and 120 edges. If 20 of the faces are triangles and 12 are pentagons, how many must be squares?

   Solution 1: Because it is a solid, it has Euler Characteristic 2, so $V - E + F = 2$. Subbing in $E = 120$ and $V = 60$, we find the total number of faces is $F = 62$. With 20 triangles and 12 pentagons, that leaves 30 faces that must be squares.

   Solution 2: Let n be the number of squares. Then we can count the number of edges: three in each triangle, four in each square and five in each pentagon to get $3(20) + 4n + 5(12) = 120 + 4n$. Since every edge is in two faces, this gives $120 + 4n = 2(120)$, which yields $n = 30$.

10. Prove that $AB - BA \neq I_n$ for any $n \times n$ matrices A and B over the real numbers, where I_n denotes the $n \times n$ identity matrix.

    Solution: We consider the sum of the elements of main diagonals. Let $A = (a_{ij})$ and $B = (b_{ij})$. Then the sum of the elements of the main diagonal of $AB - BA$ is $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} - \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji} = 0$ while the sum of the main diagonal of I_n is n. Therefore, $AB - BA \neq I_n$.

2015 Leo Schneider Student Team Competition

1. Find $\int_0^\infty \frac{1}{\sqrt{e^x-1}}\,dx$.

   Solution. Put $u = \sqrt{e^x - 1}$. Then

   $$\frac{du}{dx} = \frac{e^x}{2\sqrt{e^x-1}} = \frac{u^2+1}{2u}.$$

   Using u-substitution, our integral becomes

   $$\begin{aligned}\int_0^\infty \frac{1}{u}\cdot\frac{2u}{u^2+1}\,du &= 2\int_0^\infty \frac{1}{u^2+1}\,du \\ &= 2\arctan u\Big|_0^\infty \\ &= 2\left(\frac{\pi}{2}-0\right) \\ &= \pi.\end{aligned}$$

2. A fair die and an unfair die are in a bag. The probability of rolling a six with the unfair die is 1/4. One of the dice is randomly drawn from the bag (each one is equally likely to be chosen). A six is rolled with this die. You roll this same die again. What is the probability that a six is rolled?

   Solution. We have either chosen the fair die and need to roll another six, or we have chosen the unfair die and need to roll another six. Theoretically, if this experiment is repeated 24 times, 12 times the fair die is chosen and 12 times the unfair die is chosen. Of the 12 times the fair die is chosen, a six is rolled 2 times. Of the 12 times the unfair die is chosen, a six is rolled 3 times. Thus, the probability that the die is fair, given that a six is rolled, is 2/5. So, the probability of rolling another six is

   $$\frac{2}{5}\cdot\frac{1}{6}+\frac{3}{5}\cdot\frac{1}{4}=\frac{13}{60}.$$

   (Alternatively, we can use Bayes' Theorem to get

   $$P(\text{fair}|6) = \frac{P(6|\text{fair})P(\text{fair})}{P(6|\text{fair})P(\text{fair}) + P(6|\text{unfair})P(\text{unfair})} = \frac{\frac{1}{6}\cdot\frac{1}{2}}{\frac{1}{6}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}} = \frac{2}{5}.)$$

3. Tom and Jerry decide today that they will not pick on each other on the n-th day if $a_n = -1$ where $a_1 = 1$, $a_2 = 1$, $a_3 = -1$, and a_n for $n > 3$ is inductively given by $a_n = a_{n-1}a_{n-3}$. Will they pick on each other on the 1776-th day? Prove your answer.

   Solution:

   Direct computation shows that the first ten terms are

   $$1, 1, -1, -1, -1, 1, -1, 1, 1, -1$$

   where the last three terms are the same as the first three terms respectively. Since for $n > 3$, the a_n is defined in terms of the previous three terms, the sequence is periodic with period 7. Since $1776 = 253\cdot 7 + 5$, $a_{1776} = a_5 = -1$. Thus, Tom and Jerry will pick on each other on the 1776-th day.

4. For each positive integer k, let

$$A_k = \begin{pmatrix} 1 & k & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) Find a closed form expression for the matrix A_1^n in terms of n. Prove your answer.

(b) Find all ordered pairs (k, n) of positive integers for which $A_k^n = A_{75}$.

Solution:

In closed form, A_1^n and A_k^n are given below. Either formula can be proved by induction.

$$A_1^n = \begin{pmatrix} 1 & 2n-1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = A_{2n-1} \quad A_k^n = \begin{pmatrix} 1 & nk+n-1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} = A_{nk+n-1}$$

In order to have $A_k^n = A_{75}$ for positive integers (k, n), we must have $nk + n - 1 = 75$, or $n(k+1) = 76$. Looking at the prime factorization of $76 = 2^2 \cdot 19$, we see that there are only five possibilities: $\{(1, 38), (3, 19), (18, 4), (37, 2), (75, 1)\}$.

5. Find the limit

$$\lim_{n\to\infty} \left[\frac{(1 + \frac{1}{n})^n}{e} \right]^n$$

Solution. First, take the natural log of the n^{th} term, to get $n(n \ln(1 + \frac{1}{n}) - 1)$. The limit of this sequence is $-1/2$, which can be seen from the MacLaurin series for $\ln(1 + 1/n)$,

$$\ln(1 + 1/n) = (1/n) - \frac{(1/n)^2}{2} + \frac{(1/n)^3}{3} - \frac{(1/n)^4}{4} + \cdots$$

or by applying L'Hospital's Rule to

$$\lim_{n\to\infty} \frac{\ln(1 + \frac{1}{n}) - \frac{1}{n}}{\frac{1}{n^2}}$$

Thus the original limit is $e^{-1/2} = \dfrac{1}{\sqrt{e}}$.

6. How many solutions does the equation $20x + 15y = 2015$ have over the positive integers?

Solution. Dividing through by 5 gives us $4x + 3y = 403$. Clearly, $x = 100$ and $y = 1$ gives us a solution. Since 3 and 4 are relatively prime, every time we add 4 to y and subtract 3 from x we get another solution, and this will cover all positive integer solutions. At most, we can subtract 33 3's from 100, so there are 34 pairs of positive integer solutions.

7. A non-empty set of positive integers is said to be square-valued if the product of all of its elements is a perfect square. How many (non-empty) subsets of $\{10, 22, 30, 42, 55, 231\}$ are square-valued?

Solution:

First, factor the integers.

$$10 = 2 \cdot 5, \quad 22 = 2 \cdot 11, \quad 30 = 2 \cdot 3 \cdot 5, \quad 42 = 2 \cdot 3 \cdot 7, \quad 55 = 5 \cdot 11, \quad 231 = 3 \cdot 7 \cdot 11$$

Any product of some subset of these numbers will be a square if and only if the total exponent for each of the primes is even. This reasoning leads to the following solutions: $\{10, 22, 55\}$, $\{22, 42, 231\}$, and $\{10, 42, 55, 231\}$.

8. For the function, $f(x) = \ln\left(1 - \frac{1}{x^2}\right)$, find the value of:

$$f'(2) + f'(3) + f'(4) + \cdots + f'(2015).$$

Solution:
Using properties of logs, we rewrite $f(x) = \ln\left(\frac{x^2 - 1}{x^2}\right) = \ln(x-1) + \ln(x+1) - 2\ln x$. Then the derivative can be written as

$$f'(x) = \frac{1}{x-1} + \frac{1}{x+1} - \frac{2}{x}.$$

Then we see that the series telescopes to a sum of

$$\left(\frac{1}{2-1} - \frac{2}{2} + \frac{1}{3-1}\right) + \left(\frac{1}{2014+1} + \frac{1}{2015+1} - \frac{2}{2015}\right) = \frac{1}{2} - \frac{1}{2015} + \frac{1}{2016}$$

9. Let S be the set of vertices of a regular 36-gon. What is the smallest value of n for which any subset of S of size n must contain the three vertices of some equilateral triangle? You must prove your answer.

Solution:
Each element of S is a vertex of exactly one equilateral triangle. So S can be partitioned into 12 disjoint subsets of cardinality 3, with each cell exactly consisting of the vertices of some equilateral triangle. (In particular, if the vertices are numbered consecutively, these subsets are those containing k, $k + 12$, and $k + 24$ (mod 36).) If we choose 2 vertices from each cell, we can construct a subset of S of size 24 that does not contain the vertices of any equilateral triangle. However, if we choose any 25 vertices, there will necessarily be 3 in some cell by the Generalized Pigeonhole Principle. Therefore the answer is 25.

10. Twenty calculus students are comparing grades on their first two quizzes of the year. The class discovers that whenever any pair of students consult with one another, these two students received the same grade on their first quiz or they received the same grade on their second quiz (or both). Prove that the entire class received the same grade on at least one of the two quizzes.

Solution:
If all the students receive the same score on the first quiz, we are done. Otherwise two students, call them A and B, received different scores on the first quiz, and hence have identical scores on the second quiz. Now consider any other student C, who can't match both of A and B on the first quiz. Hence C must agree with either A or B on the second quiz, and hence also receives this common quiz two score. This logic applies to every student, so we deduce they all have the same quiz two score.

2013-2014 CONSTUM Problems and Solutions

1. Find an equation with integral coefficients whose roots include the numbers

   (a) $\sqrt{2} + \sqrt{3}$

   (b) $\sqrt{2} + \sqrt[3]{3}$

   Solution:

   (a) Write $x = \sqrt{2} + \sqrt{3}$. Then

   $$x^2 = 5 + 2\sqrt{6} \qquad \text{or} \qquad x^2 - 5 = 2\sqrt{6}.$$

   Squaring both sides, we obtain

   $$x^4 - 10x^2 + 25 = 24 \qquad \text{or} \qquad x^4 - 10x^2 + 1 = 0.$$

   (b) Let $x = \sqrt{2} + \sqrt[3]{3}$. Then

   $$x^2 = 2 + 2\sqrt{2} \cdot \sqrt[3]{3} + \sqrt[3]{9}$$

   $$x^3 = 2\sqrt{2} + 3 \cdot 2\sqrt[3]{3} + 3\sqrt{2} \cdot \sqrt[3]{9} + 3.$$

   By substituting $\sqrt[3]{3} = x - \sqrt{2}$ and

   $$\sqrt[3]{9} = x^2 - 2 - 2\sqrt{2} \cdot \sqrt[3]{3} = x^2 - 2 - 2\sqrt{2}(x - \sqrt{2}) = x^2 + 2 - 2\sqrt{2}x$$

   into the equation for x^3, we obtain

   $$x^3 = 2\sqrt{2} + 6(x - \sqrt{2}) + 3\sqrt{2}(x^2 + 2 - 2\sqrt{2}x) + 3, \text{ or}$$

   $$x^3 + 6x - 3 = \sqrt{2}(3x^2 + 2).$$

   We now square both sides to eliminate the radical and move everything to the left hand side to obtain

   $$x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1 = 0.$$

2. Let $a_1, a_2, a_3, \cdots$ be the increasing sequence of positive integers that are not divisible by 2 or 3. The sequence begins 1, 5, 7, 11, 13, 17, $\cdots$ Compute the sum of the following series.

   $$\sum_{n=1}^{\infty} \frac{1}{2^{a_n}} = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \cdots$$

   Solution. This series is absolutely convergent, so we may break it up into two series: one with the terms $a_n \equiv 1 \pmod 6$ and one with the terms that are $a_n \equiv 5 \pmod 6$. These are both geometric.

   $$\sum_{k=0}^{\infty} \frac{1}{2^{6k+1}} = \sum_{k=0}^{\infty} \frac{1}{2}\left(\frac{1}{64}\right)^k = \frac{1/2}{1 - 1/64} = \frac{32}{63}$$

   $$\sum_{k=0}^{\infty} \frac{1}{2^{6k+5}} = \sum_{k=0}^{\infty} \frac{1}{32}\left(\frac{1}{64}\right)^k = \frac{1/32}{1 - 1/64} = \frac{2}{63}$$

   So the original series converges to $\dfrac{34}{63}$.

3. It is well-known that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Find

$$\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx.$$

Solution. Using integration by parts with $u = \sin^2 x$ and $dv = x^{-2}\,dx$, followed by the substitution $w = 2x$, gives

$$\begin{aligned}\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx &= -\frac{\sin^2 x}{x}\Big|_0^\infty + \int_0^\infty \frac{\sin(2x)}{x}\,dx \\ &= \int_0^\infty \frac{\sin w}{w}\,dw \\ &= \boxed{\tfrac{\pi}{2}}.\end{aligned}$$

4. Evaluate

$$\lim_{x\to\infty} \frac{\int_0^x \sqrt{4+t^4}\,dt}{x^3}.$$

Solution:

Using L'Hospital's Rule and the Fundamental Theorem of Calculus, we have

$$\lim_{x\to\infty} \frac{\int_0^x \sqrt{4+t^4}\,dt}{x^3} = \lim_{x\to\infty} \frac{\sqrt{4+x^4}}{3x^2}.$$

Another application of L'Hospital's Rule gives

$$\begin{aligned}\lim_{x\to\infty} \frac{\sqrt{4+x^4}}{3x^2} &= \lim_{x\to\infty} \frac{x^3}{3x\sqrt{4+x^4}} \\ &= \frac{1}{3}.\end{aligned}$$

5. Let $f(n) = 25^n - 72n - 1$. Determine, with proof, the largest integer M such that $f(n)$ is divisible by M for every positive integer n.

Solution. The answer is $M = 48$. Since $f(1) = -48$, it is clear that M must be a divisor of 48. Using the factorization $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$ we may write

$$f(n) = 25^n - 1 - 72n = 24(25^{n-1} + 25^{n-2} + \cdots + 25 + 1) - 24(3n).$$

Whether n is even or odd, this is an even multiple of 24, so is divisible by 48. Thus, $M = 48$ is the largest common divisor of all the values of $f(n)$.

6. Choose N elements of $\{1, 2, 3, \ldots, 2N\}$ and arrange them in increasing order. Arrange the remaining N elements in decreasing order. Let D_i be the absolute value of the difference of the ith elements in each arrangement. Prove that $D_1 + D_2 + \cdots + D_N = N^2$.

Solution. Imagine the numbers 1 through N colored red and the numbers $N+1$ through $2N$ colored blue. The increasing arrangement will consist of some red elements followed by some blue elements. Suppose there are X red elements and so $N-X$ blue elements. The decreasing arrangement will therefore consist of X blue elements followed by $N-X$ red elements. So the ith elements of the two arrangements will consist of a "red" number paired with a "blue" number. It follows that $D_1 + D_2 + \cdots + D_N$ will be the sum of the blue numbers minus the sum of the red numbers, that is $D_1 + D_2 + \cdots + D_N = [(N+1) + (N+2) + \cdots + 2N)] - [1 + 2 + \cdots + N] = N^2$.

7. If n is a positive integer, let $r(n)$ denote the number obtained by reversing the order of the decimal digits of n. For example, $r(382) = 283$ and $r(410) = 14$. For how many two digit positive integers n is the sum of n and $r(n)$ a perfect square?

   Solution:
   Let a and b be the digits of n, so that $n = 10a + b$, $0 \le a, b \le 9$ and $a \ne 0$. Then

   $$n + r(n) = 10a + b + 10b + a = 11(a + b).$$

   Now $n + r(n)$ can only be a perfect square if 11 divides $a + b$. This leads to the following complete list of solutions:

   $$n = \{29; 38; 47; 56; 65; 74; 83; 92\}.$$

8. Determine the number of three word phrases that can be formed from the letters in MATH ALL DAY. No "words" can be empty, and words do not have to make sense. For example, MAD HAT ALLY and T DMALL YAAH are valid phrases, but not HALT MALADY. You do not have to simplify your answer.

   Solution. View the overall task of forming a three word phrase as a two step process. First, we arrange the 10 letters. Then, we decide where the two spaces go. By the Product Rule for Counting, the total number of ways to form a three word phrase will be the product of the numbers of ways to do these two smaller tasks. Because there are three A's and two L's, the number of ways to arrange the letters is $10!/(3!2!)$. Then we have $9!/(2!7!)$ ways to choose the two gaps (out of nine) where the spaces must go. So the total number of three word phrases is given by

   $$\frac{10!}{3!2!} \cdot \frac{9!}{2!7!} = 3 \cdot 10! = 10,886,400.$$

9. A circle of radius r is inscribed in a right triangle with leg $4r$. Prove that the triangle is a 3-4-5 right triangle.

   Solution. We may assume that the radius is 1 and that one leg is 4. Let the triangle be ABC with right angle at C and $AC = 4$. Let P be the point of tangency of the incircle with hypotenuse AB, let Q be the point of tangency of the incircle with leg BC, and let R be the point of tangency of the incircle with leg AC. Since the two tangents to a circle from an exterior point have the same length, we have $AP = AR = 3$. Let $BP = BQ = x$. Then $(3+x)^2 = (1+x)^2 + 4^2$. Solving, we have $x = 2$, and the result follows.

10. Recall that the set of real numbers forms an infinite-dimensional vector space V over the rationals (i.e., the set of vectors are the real numbers, and the set of scalars are the rational numbers). Let $2, 3, 5, 7, \ldots, p_{2014}$ be the first 2014 prime numbers, and define the vectors $v_k = \ln(p_k)$ for $k = 1, 2, \ldots, 2014$. Let S be the subspace of V spanned by the vectors $v_1, v_2, \ldots, v_{2014}$. Find, with proof, the dimension of S.

Solution:

The dimension is 2014, because these vectors are linearly independent. Suppose, to the contrary, that the vectors were not linearly independent. Then there exist scalars $r_1, r_2, \ldots, r_{2014}$, not all zero, such that

$$r_1 v_1 + r_2 v_2 + \cdots + r_{2014} v_{2014} = 0.$$

Multiplying by the common denominator, we get the equation

$$k_1 v_1 + k_2 v_2 + \ldots + k_{2014} v_{2014} = 0,$$

where the k_i are all integers, not all zero. Consequently, some of these k_i must be positive, and some must be negative. Ignoring the zero-value ks, and segregating the terms with positive k_i on the left, we get

$$k_{i_1} v_{i_1} + \cdots + k_{i_t} v_{i_t} = \ell_{j_1} v_{j_1} + \ldots + \ell_{j_s} v_{j_s},$$

where the ℓs are just negatives of the ks that were negative (and hence are positive). Exponentiating both sides of this equation, we get

$$p^{k_{i_1}} \cdots p^{k_{i_k}} = p^{\ell_{j_1}} \cdots p^{\ell_{j_s}}.$$

This equation shows a product using one set of primes $(p_{i_1}, p_{i_2}, \ldots, p_{i_t})$ on the left side, which is equal to a product that uses a completely different set of primes $(p_{j_1}, p_{j_2}, \ldots p_{j_s})$ on the right. But this contradicts the Fundamental Theorem of Arithmetic, which states that every integer can be factored in a unique way into primes.

2013 Leo Schneider Student Team Competition

1. For the arithmetic sequence $a_1, a_2, ..., a_{16}$, it is known that $a_7 + a_9 = a_{16}$. Find each subsequence of three terms that forms a geometric sequence.

   Solution:

   $a_{16} = a_1 + 15d$, where d is the common difference, $a_7 = a_1 + 6d$ and $a_9 = a_1 + 8d$. Since $a_7 + a_9 = a_{16}$, $a_1{+}6d{+}a_1{+}8d = a_1{+}15d$. Therefore, $d = a_1$. Therefore, the i-th term of the sequence $a_i = a_1{+}(i{-}1)d$, where $i = 1, 2, 3, ..., 16$. Since $d = a_1$, $a_i = a_1 + ia_1 - a_1 = ia_1$ so that $a_1 = 1a_1$, $a_2 = 2a_1$, $a_3 = 3a_1$,....

   Consequently, one subsequence forming a geometric sequence is a_1, a_2, a_4 (with a common ratio r=2). A second subsequence is a_1, a_3, a_9 (with a common ratio r=3). A third subsequence is a_1, a_4, a_{16} (with common ratio r=4). The fourth, fifth, and sixth subsequences are a_2, a_4, a_8 (r=2); a_3, a_6, a_{12} (r=2); a_4, a_8, a_{16} (r=2);

   For non-integer values of r such that $1 < r < 4$, we have only $r = \frac{3}{2}$ and $r = \frac{4}{3}$, so that additional sequences are a_4, a_6, a_9 and a_9, a_{12}, a_{16}

2. Compute the limit:

$$\lim_{x\to\infty} \frac{1}{xe^x} \int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt.$$

   Solution:

   Applying L'Hospital's Rule, the Fundamental Theorem of Calculus, and the Chain Rule, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{\int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt}{xe^x} &= \lim_{x\to\infty} \frac{\frac{d}{dx}\int_{x^2}^{(x+1)^2} e^{\sqrt{t}}\, dt}{\frac{d}{dx}(xe^x)} \\
&= \lim_{x\to\infty} \frac{2(x+1)e^{x+1} - 2xe^x}{xe^x + e^x} \\
&= \lim_{x\to\infty} \frac{2e(x+1) - 2x}{x+1} \\
&= 2e - 2.
\end{aligned}$$

3. Part a. Let $f(x) = e^x \sin x$. Find $f^{(10)}(0)$, the 10th derivative of f evaluated at $x = 0$.

   Part b. Let $f(x) = e^x \sin x$. Find $f^{(2013)}(0)$, the 2013th derivative of f evaluated at $x = 0$.

   Solution to part a. If $\sum_{n=0}^{\infty} a_n x^n$ is the Maclaurin series for f, then $f^{(10)}(0) = 10! \cdot a_{10}$. One way to find a_{10} is to multiply the Maclaurin series for $y = e^x$ and $y = \sin x$ together and keep track of the coefficient of x^{10}. In this way, we see that

$$a_{10} = \frac{2}{9!} - \frac{2}{3!7!} + \frac{1}{5!5!}.$$

   Thus, $f^{(10)}(0) = 10! a_{10} = \boxed{32}$.

   Solution to part b. It is easy to verify that $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, $f'''(0) = 2$, and $f^{(4)}(x) = -4f(x)$. Thus, $f^{(2013)}(0) = (-4)^{2012/4} = -2^{1006}$. (Also, this gives us another way to solve part a: $f^{(10)}(0) = 2(-4)^{8/4} = 32$.)

4. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \le 13x^2$.

   Solution:
   The condition $x^4 + 36 \le 13x^2$ is equivalent to

$$x^4 - 13x^2 + 36 = (x^2 - 9)(x^2 - 4) = (x + 3)(x - 3)(x + 2)(x - 2) \le 0,$$

   which is satisfied only on $[-3, -2]$ and on $[2, 3]$. Since $f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1) > 0$ on $[-3, -2]$ and on $[2, 3]$, the function f is increasing on both intervals. It follows that the maximum value is $\max\{f(-2), f(3)\} = 18$.

5. Let N be a positive integer containing exactly 2013 digits none of whose digits is zero. Show that N is either divisible by 2012 or N can be changed to an integer that is divisible by 2013 by replacing some but not all of its digits by zero.

   Solution:
   Let $N = b_1 b_2 \ldots b_{2013}$ be an integer where each digit $b_i > 0$ $(i = 1, 2, \ldots 2013)$. Let $N_0 = 0$, and for any $1 \le k \le 2013$, let N_k denote the number obtained by replacing all but the first k digits of N by the digit 0. By the pigeon hole principle, there exists $0 \le k_1 < k_2 \le 2013$ such that N_{k_1} and N_{k_2} are congruent modulo 2013. So the difference $N_{k_1} - N_{k_2}$ is divisible by 2013. Since $N_{k_1} - N_{k_1}$ is obtained from N by replacing some but not all of the digits of N by zero, we are done.

6. Prove that $2^{2013}+3$ is a multiple of 11.

Solution. Working mod 11, we have that

$$\begin{aligned} 2^{2013}+3 &= 2^3 \cdot 2^{2010}+3 \\ &= 8(2^5)^{402}+3 \\ &= 8(32)^{402}+3 \\ &\equiv 8(-1)^{402}+3 \\ &\equiv 0. \end{aligned}$$

7. A random number generator randomly generates integers from the set $\{1, 2, \ldots, 9\}$ with equal probability. Find the probability (with explanation) that after n numbers are generated, their product is a multiple of 10.

Solution:

The product is a multiple of 10 if an only if at least one 5 and at least one even integer have been generated. If A is the event that a 5 has been generated, and B is the event that at least one even integer has been generated, then we are looking for $\Pr(A \cap B)$. Letting E' denote the complement of an event E, we know that

$$\begin{aligned} \Pr((A \cap B)') &= \Pr(A' \cup B') \\ &= \Pr(A') + \Pr(B') - \Pr(A' \cap B'). \end{aligned}$$

The event $A' \cap B'$ represents the case in which neither a 5 nor an even integer has been generated, and consequently $\Pr(A' \cap B') = (\frac{4}{9})^n$. Since $\Pr(A') = (\frac{8}{9})^n$ and $\Pr(B') = (\frac{5}{9})^n$, it follows that

$$\Pr((A \cap B)') = \left(\frac{8}{9}\right)^n + \left(\frac{5}{9}\right)^n - \left(\frac{4}{9}\right)^n,$$

and

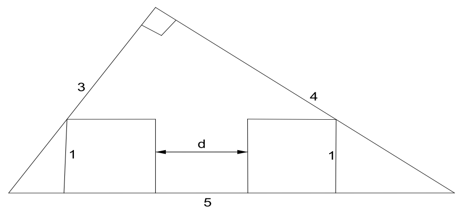
$$\Pr(A \cap B) = 1 - \Pr((A \cap B)') = 1 - \left(\frac{8}{9}\right)^n - \left(\frac{5}{9}\right)^n + \left(\frac{4}{9}\right)^n.$$

8. Planet A is going to launch n missiles at Planet B, which has n cities. Each missile will hit exactly one city. For each missile, the Planet B city that gets hit is completely random. Find the probability that exactly one city on Planet B will not get hit with any of the n missiles.

Solution:

The probability is $\frac{\binom{n}{2}n!}{n^n} = \frac{(n-1)(n-1)!}{2n^{n-2}}$. There are many ways to obtain this. Here is one. The denominator is n^n because this is the number of ways to place n missiles in n cities. The numerator is the number of ways of placing the missiles such that exactly one city is not hit. There are n ways to specify the city that does not get hit. There are $n-1$ ways of choosing the city that gets hit with two missiles. There are $\binom{n}{2}$ ways of picking the 2 missiles to hit this city. And there are $(n-2)!$ ways of placing the remaining $n-2$ missiles into the $n-2$ cities, one missile in each city. The product of these is the numerator $n(n-1)\binom{n}{2}(n-2)! = \binom{n}{2}n!$.

9. Two unit squares stand on the hypotenuse of a (3,4,5) triangle in such a way that they line inside the triangle, and a corner of one touches the side of length 3 and a corner of the other touches the side of length 4, as shown in the figure to the right. What is the distance d between the squares?



Solution:

Because the small triangle in the lower left is similar to the (3,4,5) triangle, we know that the side of length 3 in the figure has slope $\frac{4}{3}$, so the base of the small triangle in the lower left is $\frac{3}{4}$. The side of length 4 has slope $-\frac{3}{4}$, so the base of the small triangle in the lower right is $\frac{4}{3}$. Then

$$d = 5 - \frac{3}{4} - \frac{4}{3} - 2 = \frac{11}{12}.$$

10. Let A and B be 3×3 matrices with integer entries, such that $AB = A + B$. Find all possible values of $\det(A - I)$. Note: The symbol I represents the 3×3 identity matrix.

Solution:

The given equation is equivalent to

$$AB - A - B + I = I \quad \text{or to} \quad (A - I)(B - I) = I.$$

Since the matrices have integer entries, the determinants of $A - I$ and $B - I$ are integers, and the last equation implies that $\det(A - I) = \det(B - I) = \pm 1$. Both cases are possible: if $A = B = O$, then $\det(A - I) = \det(-I) = -1$, and if $A = B = 2I$, then $\det(A - I) = \det(I) = 1$.